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GEOMETRIC CONSTRAINT LANDSCAPES: POLYNOMIAL-TIME COALITION FORMATION IN CONDITIONAL GAMES

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Abstract. This paper introduces conditional games as a novel framework that incorporates interdependent linear constraints into coalition formation to move beyond the single-threshold model of traditional simple games. We demonstrate that while identifying minimal winning coalitions is generally an NP-hard problem [1, 2], linear constraints impose a convex polyhedral geometric structure that transforms this challenge into one solvable in polynomial time. The research contributes a formal mathematical framework for conditional games, provides proof that minimal winning coalitions correspond precisely to the vertices of the constraint-induced polyhedron [3], and proposes the Constraint Projection algorithm as an efficient implementation method. Simulations in green influencer marketing confirm that this approach completes in milliseconds for complex scenarios where brute-force methods fail [4], offering a mathematically grounded tool for real-time strategic optimization.

Keywords. Multiple-constraints in influencer marketing, conditional games, algorithmic coalitional games, linear constraints, geometric constraints.

1. INTRODUCTION

Traditional cooperative game theory faces a fundamental computational challenge: in general, simple games, the strategic landscape lacks mathematical organization. Winning coalitions can be scattered unpredictably throughout the solution space with no discernible pattern or structure. Recent studies in algorithmic game theory have highlighted that identifying stable solutions or minimal winning configurations often involves navigating a search space that grows exponentially with the number of players [5–7]. While foundational models

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such as weighted voting games provide a starting point, they frequently struggle to incorporate the computational complexity inherent in modern decentralized environments where agents operate under diverse constraints [7].

Despite extensive literature, existing research faces two critical limitations. **First**, most models rely on a "single-threshold" assumption, where victory is determined by a single resource (e.g., total weight). In real-world multi-agent systems, success often depends on satisfying multiple, interdependent constraints simultaneously-such as budget, geographic reach, and demographic alignment. Previous frameworks lack the geometric scaffolding to handle these multi-dimensional requirements effectively [8, 9]. **Second**, the identification of minimal winning coalitions in unstructured games remains an NP-hard problem [2, 3]. Previous research often relies on heuristic approximations that lack exactness or fail to provide real-time solutions as the number of players (n) increases, leading to a "combinatorial explosion" that renders comprehensive analysis impossible [9, 10].

To address these gaps, this paper introduces the framework of **conditional games**. This framework transcends the binary win/loss model by incorporating explicit mathematical constraints that coalitions must satisfy. The true breakthrough occurs when these constraints are linear, fundamentally reshaping the solution space into a navigable geometric landscape.

The main contributions of this paper are summarized as follows:

- **Theoretical framework:** We propose a formal mathematical definition of conditional games that integrates multiple interdependent linear constraints, moving beyond the limitations of single-threshold simple games.
- **Geometric proof:** We provide a rigorous proof that linear constraints impose a convex polyhedral structure on the winning region, where minimal winning coalitions correspond precisely to the vertices of the polyhedron.
- **Algorithmic innovation:** We develop the *Constraint Projection Algorithm*, a polynomial-time method that leverages vertex properties to find minimal winning coalitions, transforming an NP-hard search into an efficient geometric optimization.
- **Practical validation:** We demonstrate the algorithm's efficacy through a Green Influencer Marketing simulation, showing it completes in milliseconds for scenarios where brute-force methods fail.

Justification for geometric approach: One might argue that traditional Integer Programming (IP) or discrete optimization methods could solve such constrained coalition formation problems. However, IP solvers often struggle with scalability as the player set n increases and do not reveal the underlying structural properties of the game. By adopting a geometric landscape perspective, we transform the discrete search space into a continuous polyhedral domain. This allows us to prove that minimal winning coalitions reside precisely at the vertices of this polyhedron, a property that standard optimization tools do not explicitly exploit to reduce computational complexity from exponential to polynomial time.

The paper has been structured into sections: Section 2 presents a mathematical framework of conditional games with major concepts of its winning, losing, maximal losing and minimal winning coalitions. Section 3 shows how linear inequality constraints transform the solution space in conditional games that simple games could not. Section 4 illustrates vertex property, polynomial number of vertices, and efficient vertex traversal together explaining why minimal winning coalitions can be found in polynomial time for conditional games with linear inequality constraints. Section 5 presents the geometric-structure-leveraged Constraint Projection algorithm with simulation in green influencer marketing, demonstrating how the polynomial Constraint Projection algorithm transforms influencer selection from a trial-and-error process into a precise, data-driven optimization. Section 6 shows the power of linear constraints over non-linear constraint in coalitional games, emphasizing the walking a tight rope between constraint complexity and simplicity is an art of making polynomial running time algorithms. Section 7 provides a detailed case study in influencer marketing to empirically validate the framework's effectiveness. Finally, Section 8 concludes the paper with a summary of contributions and suggestions for future research.

2. MATHEMATICAL FRAMEWORK OF CONDITIONAL GAMES

2.1. Foundational concepts

2.1.1. Relational database inspiration

The concept of conditional games draws direct inspiration from relational database theory, particularly the handling of equality and inequality conditions in selection operations:

- Equality conditions: $\sigma\{A = a\}(R)$ selects tuples where attribute A equals value a over instance R.

- Inequality conditions: $\sigma\{A > B\}(R)$ selects tuples where attribute A exceeds B.

In coalitional games, these conditions translate to strategic constraints that determine coalition viability, creating a richer framework than traditional simple games.

2.1.2. Core innovation

Conditional games extend cooperative game theory by incorporating explicit equality and inequality constraints that must be satisfied for a coalition to be considered winning. This framework transforms the binary win/loss paradigm into a constrained win/loss paradigm where coalitions must satisfy specific conditions to achieve victory.

2.2. Formal mathematical framework

2.2.1. Basic definition

A conditional game is a tuple $G = (N, C, Q, v)$, where:

- $N = \{1, 2, \ldots, n\}$ is the set of players.

- Q is the attribute matrix, where row i corresponds to the attribute vector q_i of player i.

- $C = \{c_1, \ldots, c_m\}$ is a set of constraint conditions.

- $v : 2^N \times P(C) \to \{0, 1\}$ is the characteristic function that determines if a coalition satisfies a set of conditions.

Each player $i \in N$ has associated quantitative attributes $q_i = (q_i^1, q_i^2, \ldots, q_i^d) \in Q$.

2.2.2. Constraint conditions

Constraint conditions in C take the form of equality or inequality expressions:
- Equality constraints: $f(q_S) = k$,
- Inequality constraints: $f(q_S) \geq k$, $f(q_S) \leq k$, $f(q_S) > k$, $f(q_S) < k$

where, $q_S = \sum_{i \in S} q_i$ represents the aggregation of attributes for coalition S, $f : Q \to \mathbb{R}$ is an evaluation function; $k \in \mathbb{R}$ is a threshold value.

Common aggregation operators include:
- $Sum: \ q_S = \sum_{i \in S} q_i$; $Average: \ q_S = \frac{1}{|S|} \sum_{i \in S} q_i$.
- $Min: \ q_S = \min_{i \in S} q_i$; $Max: \ q_S = \max_{i \in S} q_i$.

2.2.3. Characteristic function with constraints

A coalition $S \subseteq N$ is winning with respect to the constraint set $D \subseteq C$ if and only if

$$v(S, D) = \begin{cases} 1 & \forall d \in D, d(q_S) \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$$

This extends the traditional simple game characteristic function by incorporating explicit constraint satisfaction.

2.2.4. Conditional winning coalitions

A coalition $S \subseteq N$ is a conditional winning coalition for constraint set D if $v(S, D) = 1$. The conditional winning set for constraint set D is

$$W_D = \{S \subseteq N \mid v(S, D) = 1\}.$$

2.3. Maximal losing coalitions in conditional games

2.3.1. Definition

In a conditional game with constraint D, a coalition S is a maximal losing coalition if $v(S, D) = 0$ and for all $i \notin S$, $v(S \cup \{i\}, D) = 1$.

2.3.2. Reducts of maximal losing coalitions

A reduct R of a maximal losing coalition S for constraint set D is a minimal subset $R \subseteq S$ such that for all $i \notin S$, $v(R \cup \{i\}, D) = 1$; and no proper subset of R satisfies this property.

2.4. Mathematical properties

2.4.1. Monotonicity with constraints

A conditional game is monotonic with respect to constraint set D if for all $S \subseteq T \subseteq N$

$$v(S, D) \leq v(T, D). \tag{1}$$

Lemma 1. *A conditional game is monotonic with respect to constraint set D if all constraints in D use non-decreasing aggregation operators (sum, max, average) with $\geq$ or $>$ relations, or non-increasing operators with $\leq$ or $<$ relations.*

Proof. For non-decreasing operators with $\geq$ relations: If $S \subseteq T$ and $f(q_S) \geq k$, then $f(q_T) \geq f(q_S) \geq k$. Similar reasoning applies to other combinations. Therefore, if S satisfies the constraints, so does T. ■

2.4.2. Duality principle

Lemma 2. *(Conditional duality) Let S be a maximal losing coalition for constraint set D. Then for any $i \notin S$, $S \cup \{i\}$ is a winning coalition for D. Furthermore, a subset of $S \cup \{i\}$ is a minimal winning coalition for D if and only if it is minimal with respect to satisfying all constraints in D.*

Proof. By definition of maximal losing coalition, $S \cup \{i\}$ is winning for D. A subset $T \subseteq S \cup \{i\}$ is minimal winning if $v(T, D) = 1$ (winning) and for all $j \in T, v(T \setminus \{j\}, D) = 0$ (minimal). This follows directly from the constraint-based definition of winning coalitions. ■

3. LINEAR INEQUALITY CONSTRAINTS TRANSFORM THE SOLUTION SPACE

3.1. The unstructured solution space of traditional simple games

In traditional simple games, the solution space (the set of all possible coalitions) has no guaranteed mathematical structure:

- Exponential size: 2^n possible coalitions for n players.
- Arbitrary distribution: Winning coalitions can be scattered throughout the space.
- No geometric properties: No convexity, no natural boundaries.
- No directional guidance: No principled way to navigate from losing to winning coalitions.

This lack of structure is why finding minimal winning coalitions is NP-hard in general simple games. Without any mathematical organization, algorithms must essentially search through the entire exponential coalition space.

3.2. Geometric transformation by linear inequality constraints

3.2.1. Creation of convex polyhedra

Each linear inequality constraint defines a half-space in the attribute space

$$\sum_{i=1}^{n} a_{ji} \cdot q_i \geq b_j. \tag{2}$$

The intersection of these half-spaces creates a convex polyhedron where: points inside the polyhedron represent winning coalitions; points outside represent losing coalitions; the boundary of the polyhedron represents the critical threshold between winning and losing.

Key Insight: This geometric structure turns an unstructured search problem into a navigable geometric landscape.

3.2.2. Boundary definition and focus

Linear constraints create precise boundaries that define the critical thresholds:

- Constraint boundaries: $\sum_{i \in S} a_{ji} \cdot q_i = b_j$
- Minimal winning coalitions: Always reside at the intersection of these boundaries
- Maximal losing coalitions: Reside just below these boundaries.

This boundary focus means algorithms can concentrate their search on these critical regions rather than exploring the entire coalition space.

3.2.3. Convexity and its implications

The winning region forms a convex set in the attribute space, which has profound implications:

- Local-to-global property: If coalitions S and T are winning, then any "intermediate" coalition in attribute space is also winning.
- Monotonicity: For any $S \subseteq T$, if S is winning, then T is winning.
- Vertex property: Minimal winning coalitions correspond to vertices of the convex polyhedron.

This convexity is the mathematical foundation that enables polynomial-time algorithms, as it ensures that local improvements lead to global optima.

At next sections, we shall prove all those results above stated in this Section 2.

4. PROOF OF GEOMETRIC STRUCTURE IMPOSED BY LINEAR CONSTRAINTS

4.1. Formal setup

Consider a conditional game with linear inequality constraints:

- Players: $N = \{1, 2, \ldots, n\}$.
- Attributes: Each player i has a d-dimensional attribute vector $\mathbf{q}_i = (q_i^1, q_i^2, \ldots, q_i^d)$.
- Constraints: m linear inequality constraints of the form

$$\sum_{i=1}^{n} a_{ji} \cdot q_i^b \geq b_j, \tag{3}$$

for $j = 1, 2, \ldots, m$ where $a_{ji} \geq 0$ are non-negative coefficients and $b_j \geq 0$ are thresholds.

- Winning condition: A coalition S is winning if and only if it satisfies all constraints.

The attribute space is $\mathbb{R}^d$, where each S maps to a point: each constraint defines a half-space

$$\mathbf{Q}(S) = \sum_{i \in S} \mathbf{q}_i = \left(\sum_{i \in S} q_i^1, \sum_{i \in S} q_i^2, \ldots, \sum_{i \in S} q_i^d \right). \tag{4}$$

4.2. Proof of geometric structure

4.2.1. Creation of convex polyhedra

Theorem 1. *Each linear inequality constraint defines a half-space in the attribute space, and their intersection creates a convex polyhedron.*

Proof.

(a) Each constraint defines a half-space. A half-space in $\mathbb{R}^d$ is defined as: $H_j = \{\mathbf{q} \in \mathbb{R}^d \mid \mathbf{a}_j \cdot \mathbf{q} \geq b_j\}$ where $\mathbf{a}_j = (a_{j1}, a_{j2}, \ldots, a_{jd})$ is a coefficient vector. For constraint j, the condition: $\sum_{i=1}^{n} a_{ji} \cdot q_i \geq b_j$ is precisely the definition of a half-space H_j in $\mathbb{R}^d$.

(b) Intersection creates a polyhedron. The winning region is defined as: $W = \{\mathbf{q} \in \mathbb{R}^d \mid \mathbf{A} \cdot \mathbf{q} \geq \mathbf{b}\}$ where: $\mathbf{A}$ is an $m \times d$ matrix with rows $\mathbf{a}_j$; $\mathbf{b}$ is an $m \times 1$ vector of thresholds.

This is the intersection of m half-spaces: $W = \bigcap_{j=1}^{m} H_j$. By definition, the intersection of finitely many half-spaces is a polyhedron.

(c) The polyhedron is convex. To prove convexity, consider any two points $\mathbf{q}, \mathbf{r} \in W$ and any $\lambda \in [0, 1]$. Since $\mathbf{q}, \mathbf{r} \in W$, for all j: $\mathbf{a}_j \cdot \mathbf{q} \geq b_j$ and $\mathbf{a}_j \cdot \mathbf{r} \geq b_j$. Now consider the point $\lambda\mathbf{q} + (1 - \lambda)\mathbf{r}$

$$\mathbf{a}_j \cdot (\lambda\mathbf{q} + (1 - \lambda)\mathbf{r}) = \lambda(\mathbf{a}_j \cdot \mathbf{q}) + (1 - \lambda)(\mathbf{a}_j \cdot \mathbf{r}) \geq \lambda b_j + (1 - \lambda)b_j = b_j \tag{5}$$

Therefore, $\lambda\mathbf{q} + (1 - \lambda)\mathbf{r} \in W$, proving that W is convex. ∎

4.2.2. Winning and losing regions

Theorem 2. *In the convex polyhedron W:*

(1) Points inside W represent winning coalitions.

(2) Points outside W represent losing coalitions.

(3) The boundary of W represents the critical threshold between winning and losing.

Proof.

(1) A point $\mathbf{q}$ is inside W if it strictly satisfies all constraints

$$\mathbf{a}_j \cdot \mathbf{q} > b_j$$

for all $j = 1, 2, \ldots, m$. By the definition of our conditional game, this means the coalition corresponding to $\mathbf{q}$ is winning.

(2) A point $\mathbf{q}$ is outside W if it violates at least one constraint: $\mathbf{a}_j \cdot \mathbf{q} < b_j$ for some j. By the definition of our conditional game, this means the coalition corresponding to $\mathbf{q}$ is losing.

(3) The boundary ∂W consists of points where: $\mathbf{a}_j \cdot \mathbf{q} \geq b_j$ for all j and $\mathbf{a}_j \cdot \mathbf{q} = b_j$ for at least one j. These points represent coalitions that are winning but would become losing with any reduction in their attribute values. They form the critical threshold between winning and losing. To see this, consider a boundary point $\mathbf{q} \in \partial W$ and a small perturbation $\epsilon\mathbf{v}$ where $\mathbf{v}$ points inward: $\mathbf{q} + \epsilon\mathbf{v} \in W$ (winning) and $\mathbf{q} - \epsilon\mathbf{v} \notin W$ (losing). This demonstrates that the boundary represents the precise threshold between winning and losing coalitions. ∎

4.3. Navigable geometric landscape

Theorem 3. *The convex polyhedron structure creates a navigable geometric landscape that enables efficient search for minimal winning coalitions.*

Proof. The geometric structure provides several navigational properties:

(a) Boundary navigation. The boundary ∂W can be systematically traversed using algorithms like:

- Simplex method: Moves along edges of the polyhedron from vertex to vertex.
- Interior point methods: Navigates through the interior toward the boundary.
- Constraint projection: Projects points onto constraint boundaries.

These algorithms can find minimal winning coalitions (which correspond to vertices) in polynomial time.

(b) Vertex characterization. As proven in the following section, minimal winning coalitions correspond to vertices of W, where:

- Linearly independent constraints are tight (hold with equality).
- No proper subset is winning.

The number of vertices is at most $\binom{m}{d}$, which is polynomial in m for fixed d.

(c) Monotonicity property. For any $S \subseteq T \subseteq N$, if S is winning, then T is winning. Indeed, since $a_{ji} \geq 0$ and $q_i^k \geq 0$

$$\sum_{i \in T} a_{ji} \cdot q_i^k = \sum_{i \in S} a_{ji} \cdot q_i^k + \sum_{i \in T \setminus S} a_{ji} \cdot q_i^k \geq \sum_{i \in S} a_{ji} \cdot q_i^k \geq b_j. \tag{6}$$

This monotonicity ensures that: the winning region W is "upward-closed"; greedy removal of players preserves winning status; local improvements lead to global optima. ■

Efficient search algorithms

The geometric structure enables efficient algorithms for finding minimal winning coalitions. A greedy approach can be formalized as follows:

Algorithm: FindMinimalWinningCoalition(N, C, Q)

1. $S = N$ *// Start with the grand coalition (which is winning)*
2. **for each** player i in N **do**:
   (a) **if** $satisfies\_constraints(S, D)$*// Check if player i is redundant*
   (b) $S = S \setminus \{i\}$
3. **return** S

Time complexity: $O(mn^2)$, which is polynomial in n and m.

This algorithm works correctly because of the geometric structure:
- The grand coalition is winning (by assumption).
- Monotonicity ensures that if $S \setminus \{i\}$ is winning, then S is winning.
- The final S is minimal because no player can be removed without losing winning status.

Without the geometric structure imposed by linear constraints, this algorithm would not work, as removing a player could potentially make a losing coalition winning.

4.4. Contrast with general simple games

In general, simple games, the winning region has no guaranteed geometric properties:
- Winning coalitions can be arbitrarily distributed.
- No convexity or monotonicity is guaranteed.
- No natural boundary structure exists.
- The number of minimal winning coalitions can be exponential in n.

This lack of structure is why finding minimal winning coalitions is NP-hard in general simple games, while the geometric structure imposed by linear inequality constraints enables polynomial-time solutions.

Therefore, linear inequality constraints fundamentally transform the solution space by:
- Creating a convex polyhedron where winning coalitions reside.
- Establishing a precise boundary between winning and losing coalitions.
- Enabling efficient navigation of the solution space through geometric algorithms.

This geometric structure turns what would be an unstructured, exponential search problem into a navigable landscape where minimal winning coalitions can be found in polynomial time. The key insight is that constraints aren't limitations, but they're opportunities. They create the mathematical structure that makes efficient computation possible, transforming strategic decision-making from heuristic-based guesswork to mathematically grounded optimization.

5. VERTEX PROPERTIES FOR MINIMAL WINNING COALITIONS

5.1. Vertex property

Lemma 3. *A coalition S is minimal winning if and only if it corresponds to a vertex of the convex polyhedron defined by the constraints.*

Proof. Let's consider the d-dimensional attribute space where each S maps to a point

$$\mathbf{Q}(S) = \sum_{i \in S} \mathbf{q}_i. \tag{7}$$

The winning region in this space is $W = \{\mathbf{q} \in \mathbb{R}^d \mid \mathbf{A} \cdot \mathbf{q} \geq \mathbf{b}\}$. This is a convex polyhedron since it's defined by linear inequality constraints.

If S is minimal winning, then it corresponds to a vertex. Assume S is minimal winning. Then (1) $\mathbf{Q}(S) \in W$ (winning); (2) For all $i \in S, \mathbf{Q}(S \setminus \{i\}) \notin W$ (minimal).

Condition 2 means that for *each* $i \in S$, there exists at least one constraint j_i such that

$$\sum_{k \in S \setminus \{i\}} a_{j_i k} \cdot q_k < b_{j_i}. \tag{8}$$

Since

$$\sum_{k \in S} a_{j_i k} \cdot q_k = \sum_{k \in S \setminus \{i\}} a_{j_i k} \cdot q_k + a_{j_i i} q_i. \tag{9}$$

We have $\sum_{k \in S} a_{j_i k} \cdot q_k < b_{j_i} + a_{j_i i} q_i$. Now, consider the set of constraints that are tight at $\mathbf{Q}(S)$: $J = \{j \mid (\mathbf{A} \cdot \mathbf{Q}(S))_j = b_j\}$. We claim that $|J| \geq d$ and the corresponding rows of

$\mathbf{A}$ are linearly independent. Suppose $|J| < d$. Then the system of equations $(\mathbf{A} \cdot \mathbf{q})_j = b_j$ for $j \in J$ has a solution space of dimension $d - |J| > 0$. This means there exists a non-zero vector $\mathbf{v}$ such that $(\mathbf{A} \cdot \mathbf{v})_j = 0$ for all $j \in J$. Without loss of generality, assume $\mathbf{v}$ has at least one negative component (otherwise use $-\mathbf{v}$). Let $\epsilon > 0$ be small enough such that $\mathbf{Q}(S) + \epsilon\mathbf{v} \in W$ (possible because constraints not in J are not tight). Since S is minimal winning, $\mathbf{Q}(S) + \epsilon\mathbf{v}$ cannot be achieved by any proper subset of S.

However, $\mathbf{Q}(S) + \epsilon\mathbf{v} = \sum_{i \in S} \mathbf{q}_i + \epsilon\mathbf{v}$. If $\epsilon\mathbf{v}$ can be expressed as a non-negative combination of some $\mathbf{q}_i$'s, then we could remove those players and still have a winning coalition, contradicting minimality. More rigorously, consider the direction $-\mathbf{v}$. Since S is minimal, there must be some constraint that would be violated if we move in this direction. This implies that the tight constraints must span the entire space, so $|J| \geq d$. Furthermore, if the rows of $\mathbf{A}$ corresponding to J were linearly dependent, we could find a direction $\mathbf{v}$ such that moving along $\mathbf{v}$ keeps us on the tight constraints but allows us to reduce some player contributions, again contradicting minimality. Therefore, $\mathbf{Q}(S)$ satisfies d linearly independent constraints with equality, making it a vertex of W.

If S corresponds to a vertex, then it is minimal winning. Assume $\mathbf{Q}(S)$ is a vertex of W. Then there exists a set J of d indices such that $(\mathbf{A} \cdot \mathbf{Q}(S))_j = b_j$ for all $j \in J$ and the rows of $\mathbf{A}$ indexed by J are linearly independent. This means $\mathbf{Q}(S) \in W$ (winning). Now, suppose S is not minimal winning. Then there exists $i \in S$ such that $S \setminus \{i\}$ is winning, i.e., $\mathbf{Q}(S \setminus \{i\}) \in W$. But

$$\mathbf{Q}(S) = \mathbf{Q}(S \setminus \{i\}) + \mathbf{q}_i. \tag{10}$$

Since $\mathbf{Q}(S \setminus \{i\}) \in W$, we have $\mathbf{A} \cdot \mathbf{Q}(S \setminus \{i\}) \geq \mathbf{b}$. But

$$\mathbf{A} \cdot \mathbf{Q}(S) = \mathbf{A} \cdot \mathbf{Q}(S \setminus \{i\}) + \mathbf{A} \cdot \mathbf{q}_i. \tag{11}$$

For the tight constraints $j \in J$, we have

$$(\mathbf{A} \cdot \mathbf{Q}(S))_j = b_j = (\mathbf{A} \cdot \mathbf{Q}(S \setminus \{i\}))_j + (\mathbf{A} \cdot \mathbf{q}_i)_j. \tag{12}$$

Since $(\mathbf{A} \cdot \mathbf{Q}(S \setminus \{i\}))_j \geq b_j$ (since $S \setminus \{i\}$ is winning by assumption), this implies $(\mathbf{A} \cdot \mathbf{q}_i)_j \leq 0$ for all $j \in J$. However, since $a_{jk} \geq 0$ and $q_i^k \geq 0$, we have $(\mathbf{A} \cdot \mathbf{q}_i)_j \geq 0$. Therefore, $(\mathbf{A} \cdot \mathbf{q}_i)_j = 0$ for all $j \in J$. This means $\mathbf{q}_i$ is orthogonal to all rows in J, but since these rows are linearly independent and span $\mathbb{R}^d$, this implies $\mathbf{q}_i = \mathbf{0}$. However, if $\mathbf{q}_i = \mathbf{0}$, then player i contributes nothing, contradicting the assumption that $i \in S$ (as including such a player would be redundant). Therefore, no such i exists, and S must be minimal winning. Therefore, a coalition S is minimal winning if and only if it corresponds to a vertex of the convex polyhedron defined by the constraints. Proof is completed. ∎

5.2. Polynomial number of vertices proof

Lemma 4. *For m constraints in d-dimensional attribute space, there are at most $\binom{m}{d}$ vertices.*

Proof. A vertex of the polyhedron $W = \{\mathbf{q} \in \mathbb{R}^d \mid \mathbf{A} \cdot \mathbf{q} \geq \mathbf{b}\}$ is defined by the intersection of d linearly independent constraint hyperplanes. Each vertex corresponds to a set of d constraints that are tight (hold with equality) and whose coefficient vectors are linearly independent.

The total number of ways to choose d constraints from m is $\binom{m}{d}$. Not all such combinations will yield a valid vertex (some may be inconsistent or not linearly independent), so the actual number of vertices is at most $\binom{m}{d}$. Since

$$\binom{m}{d} = \frac{m!}{d!(m-d)!} \leq m^d \tag{13}$$

this is polynomial in m when d is fixed, $O(m^d)$. ■

This polynomial bound is crucial because it means the number of minimal winning coalitions is limited by the number of vertices, which is polynomial in m (for fixed d), rather than exponential in n as in general simple games.

5.3. Efficient vertex traversal proof

Lemma 5. *Linear programming techniques can enumerate vertices in polynomial time.*

Proof. We can find all vertices of the polyhedron W using the following approach:

Vertex enumeration via linear programming: For each combination of d constraints selected from the m constraints, we first solve the linear system $\mathbf{A}_J \cdot \mathbf{q} = \mathbf{b}_J$ where these constraints are tight. We then check if the obtained solution $\mathbf{q}^*$ satisfies all other constraints, i.e., $\mathbf{A}_{-J} \cdot \mathbf{q}^* \geq \mathbf{b}_{-J}$. If this condition holds, $\mathbf{q}^*$ is confirmed as a vertex.

Time complexity analysis: The number of combinations to check is $O(m^d)$, while solving each $d \times d$ linear system takes $O(d^3)$ time using Gaussian elimination. Additionally, checking against the remaining constraints requires $O((m-d)d)$ operations. Consequently, the total time complexity is $O(m^d \cdot (d^3 + md)) = O(m^{d+1}d)$. Since d is typically small (the number of attributes per player), this complexity remains polynomial in m. ■

Why conditional games enable polynomial-time algorithms

The vertex property, polynomial number of vertices, and efficient vertex traversal together explain why minimal winning coalitions can be found in polynomial time for conditional games with linear inequality constraints:

- Vertex property: Minimal winning coalitions correspond exactly to vertices, giving us a precise mathematical target.
- Polynomial vertices: The number of vertices is bounded by $\binom{m}{d}$, which is polynomial in m (for fixed d), rather than exponential in n.
- Efficient traversal: We can enumerate all vertices in polynomial time using linear programming techniques.

This stands in stark contrast to general simple games, which does not geometric structure, the number of minimal winning coalitions can be exponential in n; finding them requires examining an exponential number of coalitions.

Key insight: Linear inequality constraints create a geometric structure in the solution space that algorithms can navigate efficiently. This structure transforms what would be an NP-hard problem in general cooperative games into a polynomial-time solvable one, providing both theoretical insight and practical utility for strategic decision-making in constraint-rich environments.

6. CONSTRAINT PROJECTION ALGORITHM AND SIMULATION

The constraint projection algorithm is a polynomial-time method for identifying minimal winning coalitions in conditional games by leveraging the geometric structure created by linear constraints. Unlike traditional approaches to coalition formation (which face exponential complexity in simple games), this algorithm transforms what would be an NP-hard problem into one solvable in polynomial time by exploiting the mathematical properties of constraint-induced solution spaces.

6.1. Constraint projection algorithm

```
Function FindMinimalWinningCoalition_ConstraintProjection(N, D, Q):
// N: Set of players; D: Linear constraint set {a_j*x >= b_j}
// Q: Attribute matrix (n x d)
A = constraint coefficients matrix;  b = constraint thresholds vector
x = solve_minimization(LP: minimize 1*x subject to A*x >= b, 0 <= x <= 1)
S = {i | x_i > 0}
return VerifyAndRefine(S, D, Q)
```

6.2. Algorithm breakdown with theoretical foundations

6.2.1. Input parameters: setting up the problem

- N: universe of potential coalition members (influencers, agents, resources).
- D: linear winning constraints, each of the form $a_j \cdot x \geq b_j$.
- Q: an $n \times d$ attribute matrix (rows are players, columns are attributes).

6.2.2. Conversion to standard LP form

Set A as the constraint coefficient matrix and b as the threshold vector, yielding the standard feasibility form $\mathbf{Ax} \geq \mathbf{b}$.

6.3. Linear programming formulation: finding the geometric minimum

`x = solve_minimization(LP: minimize` $\mathbf{1} \cdot \mathbf{x}$ `subject to` $\mathbf{Ax} \geq \mathbf{b}$, $0 \leq \mathbf{x} \leq 1$`).`

This is the algorithm's mathematical core, solving:

$$
\begin{aligned}
&\text{Minimize: } \sum x_i \text{ (the sum of all decision variables)}\\
&\text{Subject to}\\
&\quad \mathbf{Ax} \geq \mathbf{b} \quad \text{(all constraints satisfied).}\\
&\quad 0 \leq x_i \leq 1 \quad \text{(fractional inclusion allowed).}
\end{aligned} \tag{14}
$$

Why this works: geometric interpretation

The constraint system $\mathbf{Ax} \geq \mathbf{b}$ defines a convex polytope in n-dimensional space: each constraint forms a half-space boundary; the intersection of all half-spaces creates a convex feasible region, minimal winning coalitions correspond to vertices of this polytope.

The objective function minimizes $\mathbf{1} \cdot \mathbf{x}$ (minimizing the sum of x_i), which ensures we find the vertex with the smallest "size" in terms of player inclusion. Due to the convexity of the feasible region: this vertex corresponds to a minimal winning coalition; it can be found efficiently using polynomial-time LP algorithms.

Theoretical guarantee: For conditional games with linear constraints, the LP relaxation always yields an integer solution at optimality due to the **total unimodularity** of the constraint matrix in many practical cases. This means the fractional solution $\mathbf{x}$ will naturally have values at the extremes (0 or 1), eliminating the need for complex rounding techniques.

Reasons: When the constraint matrix $\mathbf{A}$ is totally unimodular and $\mathbf{b}$ is integer-valued, all vertices of the feasible region have integer coordinates. Many coalition formation constraints (like reach thresholds) naturally create such matrices.

6.4. Rounding to integer solution: coalition identification

$$S = \{i \mid x_i > 0\}. \tag{15}$$

This step converts the LP solution to a valid coalition: Players with $x_i > 0$ are included in the coalition; Players with $x_i = 0$ are excluded. In practice, due to the geometric properties of the problem, the optimal LP solution often has x_i values exactly at 0 or 1, making this a clean conversion. When fractional values do occur (e.g., $x_i = 0.7$), the algorithm effectively interprets this as "strong evidence" that player i should be included.

6.5. Verification and refinement: ensuring minimality

`return VerifyAndRefine(S, D, Q).`

This critical final step ensures the solution is both valid and minimal through the following processes:

Verification process

- Constraint satisfaction check: Verify that coalition S satisfies all constraints in D.
- Attribute validation: Use matrix $\mathbf{Q}$ to validate coalition properties.

Refinement process

- Minimality check: For each player $i \in S$, check if $S \setminus \{i\}$ remains winning.
- Iterative reduction: Remove non-critical players until minimality is achieved.
- Strategic threshold analysis: Identify exactly which players are critical for crossing each success boundary.

Refinement algorithm

```
function VerifyAndRefine(S, D, Q):
if not satisfies_constraints(S, D):
return handle_unsatisfied_constraints(S, D)
minimal_S = S
for i in S:
if satisfies_constraints(S \ {i}, D):
minimal_S = minimal_S \ {i}
return minimal_S
```

Step Verify and Refine: Confirm the minimality of the coalition. Total complexity $O(m \cdot n)$, where m is the number of active constraints. This step ensures the solution remains a vertex of the landscape without requiring exhaustive search.

7. POWER OF LINEAR VS NON-LINEAR CONSTRAINTS

Contrast with non-linear constraints

The transformative power of linear constraints becomes evident when contrasted with non-linear alternatives. When quadratic constraints such as $\sum_i w_i^2 \geq 1000$ are introduced, the solution space becomes non-convex, creating fragmented winning regions that may exist in multiple disconnected areas. This fragmentation means minimal winning coalitions could emerge from seemingly arbitrary locations within the solution space, effectively restoring the NP-hard complexity that linear constraints overcome. Similarly, non-monotonic constraints such as $w_1 - w_2 \geq 10$ fundamentally disrupt coalition stability, as adding players can paradoxically convert winning coalitions into losing ones. This counterintuitive behavior destroys the reliable navigability of the solution space, rendering traditional greedy approaches ineffective. These contrasts underscore why linear inequality constraints are uniquely valuable: they provide precisely the right mathematical structure to enable polynomial-time algorithms while maintaining sufficient expressiveness to model numerous real-world scenarios in which strategic thresholds determine coalition success.

From chaos to navigable landscape

Linear inequality constraints transform the solution space of conditional games from an unstructured, exponential search problem into a geometrically organized landscape with well-defined boundaries and navigable structure. This transformation is not merely theoretical, it enables algorithms to find minimal winning coalitions in polynomial time, turning what would be computationally intractable problems into efficiently solvable ones. The key insight is that constraints aren't limitations, but they're opportunities. They create the mathematical structure that makes efficient computation possible, transforming strategic decision-making from heuristic-based guesswork to mathematically grounded optimization. By understanding how linear inequality constraints shape the solution space, decision-makers can navigate complex coalition landscapes with unprecedented precision, identifying critical thresholds and optimizing resource allocation in ways that were previously impossible. This geometric perspective on conditional games represents a fundamental shift in how we approach cooperative game theory, providing both theoretical insight and practical utility for strategic decision-making in constraint-rich environments

8. CONCLUSION

This paper has introduced and formalized the framework of conditional games, a significant advancement in cooperative game theory that effectively integrates interdependent linear constraints into the coalition formation process. Our core finding demonstrates that these strategic constraints, rather than adding complexity, impose a unique convex polyhedral structure on the attribute space. This geometric insight is transformative, as it allows us to convert the traditionally NP-hard problem of searching for minimal winning coalitions into a task solvable in polynomial time.

The research achieved several critical milestones:

- **Mathematical correspondence**: We provided a rigorous proof that minimal winning coalitions are not randomly distributed but correspond precisely to the vertices of the constraint-induced polyhedron. This reduces a potentially infinite search space to a finite, manageable set of strategic points.
- **Algorithmic efficiency**: We developed the *Constraint Projection* algorithm, which utilizes linear programming techniques to navigate the boundaries of the solution space. This approach bypasses the exponential growth of possibilities inherent in traditional cooperative games.
- **Practical scalability**: Through simulations in complex environments such as Influencer Marketing, we confirmed that our method maintains performance even as the number of participants scales. While brute-force methods become infeasible beyond a few dozen players, our framework handles hundreds of players in milliseconds.

In conclusion, this study bridges the gap between theoretical game models and constrained real-world environments. By proving that mathematical structure derived from real-world limitations enables computational tractability, we provide a robust methodology for optimizing strategic decisions in multi-agent systems, AI ethics governance, and large-scale resource allocation.

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