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Multi-material proportional topology optimization using hybrid threshold interpolation

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Abstract. In this paper, Proportional Topology Optimization (PTO) is employed to solve multi-material topology optimization problems, considering the minimum compliance problems satisfied by the mass and cost constraints. A hybrid interpolation using threshold functions is applied to the elastic modulus, while the cost is represented as a linear function. Functions with scaling and translation coefficients are introduced to interpolate the elastic modulus and cost properties for multiple materials with respect to normalized density variables. The numerical examples are conducted from multiple perspectives to illustrate the proposed method. Studies on filter radius, projection parameter, hybrid coefficient, and initial design variables reveal their influence on the optimal solution of the problem, accompanied by a real-world example involving steel and aluminum materials to demonstrate the impact of cost constraints.

Keywords: Topology optimization, proportional topology optimization, minimum compliance problem, multi-material design.

Classification numbers: 5.4.3, 5.6.2.

1. INTRODUCTION

Topology optimization in solid mechanics is a powerful engineering methodology that has revolutionized the design and analysis of structures and components. It is a computational technique aimed at optimizing the distribution of material within a given design domain to achieve the best possible performance, often with constraints on factors like mass, stiffness, or cost. In recent years, there has been a notable increase in the number of studies focusing on the optimization of multi-material structures. The Solid Isotropic Material with Penalization (SIMP) approach has gained popularity due to its practicality and conceptual clarity. Pioneering works in this domain include those by Sigmund [1] and Bendsøe *et al*. [2]. Zuo and Saitou [3] introduced an innovative approach known as ordered multi-material SIMP interpolation. In this approach, material properties of different candidate materials are expressed as functions of ordered normalized mass densities. López *et al*. [4] presented a comprehensive set of methodologies aimed at addressing various aspects of multi-material topology optimization. Silvera *et al*. [5]

introduced a fresh material model for ordered SIMP, with the aim of improving the method Zou proposed.

Complementing these gradient-based methods, there are gradient-free approaches, such as PTO introduced by Biyikli and To [6], which allocate material proportionally based on strain energy without requiring sensitivity calculations. In addition, various non-derivative methods have been developed and applied in numerous topology optimization problems. Cui *et al*. [7] extended the PTO to address multi-material problems with a modified interpolation scheme. Nguyen *et al*. [8] presents an extension for the PTO algorithm to solve multi-material topology optimization of compliant mechanism problems.

The aim of this research is to create a new computational scheme for optimizing multi-material structures, using PTO and a novel interpolation function for elastic modulus. The threshold function is chosen due to its practical compatibility with real-world manufacturing processes. Even though design variable values diverge significantly from material density values, threshold function effectively narrows the elastic modulus values towards the true material's ones. This helps the model become more realistic. The remainder of this paper is organized as follows. In Section 2, novel ideas on the interpolation of the elastic modulus using threshold functions, the distribution of the cost function, and the density projection technique are presented. All of these concepts are integrated into the PTO method to offer a solution approach for multi-material problems. In Section 3, several numerical examples are shown to demonstrate the effectiveness and feasibility of the proposed method in designing common multi-material models. Conclusions and comments are drawn in the last section.

2. MAIN IDEAS FOR THE STUDY

In this section, the minimum compliance problem for multi-materials with constraints on both mass and cost is introduced. Next, we discuss interpolation techniques for the elastic modulus and cost variables in the context of multi-material problems. Additionally, in order to concentrate density values toward the candidate material density, it is essential to explore the implementation of density projections. Finally, we present the PTO algorithm for multi-material problems.

2.1. Topology optimization problem formulation

Considering the presence of multiple materials characterized by three properties: density, elastic modulus and cost; we formulate the topology optimization problem based on density. The primary objective is to minimize compliance while adhering to constraints associated with structural mass and cost. Cost constraint plays a vital role in real-world production, influencing material choices and structural designs to ensure that optimized solutions meet both performance objectives and economic feasibility. This problem can be expressed as follows:

$$\begin{aligned}
&\min_{x}: \; c(\mathbf{x}) = \mathbf{U}^T\mathbf{K}\mathbf{U} = \sum_{e=1}^{N} E_e\left(x_e\right)\mathbf{u}_e^T\mathbf{k}_0\mathbf{u}_e \\
&s.t.: \quad \mathbf{K}\mathbf{U} = \mathbf{F} \\
&\qquad M(\mathbf{x})/M_0 \le f_M, \; M(\mathbf{x}) = \sum_{e=1}^{N} V_e x_e \\
&\qquad C(\mathbf{x})/C_0 \le f_C, \; C(\mathbf{x}) = \sum_{e=1}^{N} V_e x_e C_e \\
&\qquad \mathbf{0} \le \mathbf{x} \le \mathbf{1}
\end{aligned} \tag{1}$$

where c is structure compliance, $\mathbf{K}$ is the global stiffness matrix, $\mathbf{U}$ and $\mathbf{F}$ are the global displacement vector and force vector, respectively; $\mathbf{k}_0$ is the element stiffness matrix for an element with unit modulus of elasticity, $\mathbf{u}_e$ is the element displacement vector, E_e is material elastic modulus of the e-th element, x_e is the element density and $\mathbf{x}$ is the vector of design variables which contains x_e, N is the number of element used to discretize the design domain, M and C are the mass and the cost of the current design, respectively, M_0 and C_0 are the mass and the cost of the design domain fully filled with the heaviest material, respectively, f_M and f_C are the allowed mass and cost fraction, respectively; V_e and C_e are the volume and cost of the e-th element.

2.2. Interpolation for elastic modulus and cost function

The function $E_e(x_e)$ in equation (1) is approximated by a threshold function proposed by Wang *et al*. [9], which takes the following form:

$$E_e^T(x_e) = \begin{cases} \dfrac{E_{i+1}-E_i}{2}\left(e^{-\beta A_e(x_e)} - A_e(\rho_e)e^{-\beta}\right) + E_i & \text{for } x_e \in [\rho_i, \rho_{i+1/2}] \\ \dfrac{E_{i+1}-E_i}{2}\left(1-e^{-\beta B_e(x_e)} + B_e(\rho_e)e^{-\beta}\right) + \dfrac{E_{i+1}+E_i}{2} & \text{for } x_e \in [\rho_{i+1/2}, \rho_{i+1}] \end{cases} \tag{2}$$

where

$$A_e(x_e) = \frac{\rho_{i+1/2} - x_e}{\rho_{i+1/2} - \rho_i}, \quad B_e(x_e) = \frac{x_e - \rho_{i+1/2}}{\rho_{i+1} - \rho_{i+1/2}} \tag{3}$$

and E_i and E_{i+1} are the elastic modulus of ordered candidate material i and $i+1$, respectively. The threshold $\rho_{i+1/2}$ is the middle point between two phases. It leads to $\rho_{i+1/2} = (\rho_i + \rho_{i+1})/2$ where ρ_i and ρ_{i+1} are the density of ordered candidate material i and $i+1$, respectively.

A common interpolation for elastic modulus is a power function. For multi-material problems, Zou [3] proposed an extended power function as

$$E_e^Z(x_e) = \frac{E_{i+1}-E_i}{\rho_{i+1}^P - \rho_i^P}\left(x_e^P - \rho_i^P\right) + E_i \quad \text{for } x_e \in [\rho_i, \rho_{i+1}] \tag{4}$$

where p is referred to the penalty parameter.

To investigate the impact of the elastic modulus on the results, we introduce a hybrid function combining a power function and a threshold function as follows:

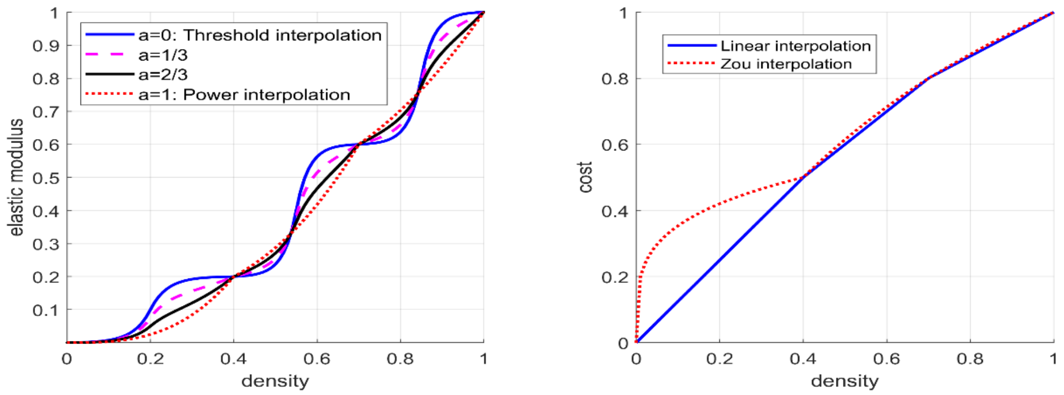
$$E_e(x_e) = (1-a)E_e^T(x_e) + aE_e^Z(x_e) \tag{5}$$

where a is the hybrid coefficient. When $a=0$, $E_e(x_e)$ adopts the threshold function. When $a=1$, $E_e(x_e)$ takes on the form of a power function (see Figure 1-a).

When employing the PTO method, there is no need for an overly complex cost function. We propose utilizing a linear function that aligns with practicality (see Figure 1-b).

$$C_e(x_e) = \frac{C_{i+1}-C_i}{\rho_{i+1}-\rho_i}(x_e - \rho_i) + C_i \text{ for } x_e \in [\rho_i, \rho_{i+1}] \tag{6}$$

where C_i and C_{i+1} are the cost of ordered candidate material i and $i+1$, respectively.



a) Elastic modulus interpolation b) Cost interpolation

Figure 1. Elastic modulus and cost interpolation.

2.3. Filtering and Projection for density

In order to mitigate the emergence of checkerboard patterns in the density distribution, a widely adopted method is to employ a density filter. Silveira and Palma [5] proposed the density filter that modifies the design variable as follows:

$$\tilde{x}_e = \frac{1}{\sum_{i \in N_e} H_{ei}} \sum_{i \in N_e} H_{ei} x_i, \qquad H_{ei} = \max\left(0, r - d(e,i)\right) \tag{7}$$

where r is the filter radius, $d(e,i)$ is distance from element e to element i and N_e is the set of elements i for which the distance $d(e,i)$ is shorter than r.

While addressing multi-material optimization problems, numerous intermediate density elements emerge within the final structural configuration. These intermediate elements may exist between lighter solid materials and void regions or between different candidate solid materials. To tackle this challenge, the threshold projection technique is once again employed. As a result, the final density has the following form:

$$\bar{x}_e = \begin{cases} \dfrac{\rho_{i+1} - \rho_i}{2}\left(e^{-bA_e(\tilde{x}_e)} - A_e(x_e)e^{-b}\right) + \rho_i & \text{for } \tilde{x}_e \in \left[\rho_i, \rho_{i+1/2}\right] \\ \dfrac{\rho_{i+1} - \rho_i}{2}\left(1 - e^{-bB_e(\tilde{x}_e)} + B_e(x_e)e^{-b}\right) + \dfrac{\rho_{i+1} + \rho_i}{2} & \text{for } \tilde{x}_e \in \left[\rho_{i+1/2}, \rho_{i+1}\right] \end{cases} \tag{8}$$

where A_e and B_e are introduced in equation (3), b is a projection parameter whose value incrementally grows during the iterations of the optimization process.

2.4. The optimal algorithm

The multi-material optimization process is conducted as follows. Initially, we set an initial density value, denoted as $\mathbf{x}_{(0)}$, for the design variable $\mathbf{x}$, marking the starting point at $k = 0$ within the computational loop. This design variable $\mathbf{x}$ is then utilized for elastic modulus interpolation and compliance value calculation. Subsequently, we redesign the density variable using the PTO method, prioritizing compliance as the primary comparison criterion. To make the results better, we apply a density projection that concentrates the density variable towards the density of candidate materials, thereby producing an updated design variable at (k + 1)-th iteration. The deviation between the design variables $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(k+1)}$, combined with compliance and cost constraints, is used to evaluate the convergence of the method. If the two design variables exhibit significant differences or fail to satisfy the imposed constraints, we once again utilize the recently obtained design variable for elastic modulus interpolation, compliance calculation, the PTO implementation, and density projection utilization to determine a new design variable. This iterative process continues until the values of the design variables converge, and all constraints are simultaneously satisfied. The flowchart outlining this methodology can be found in Figure 2.

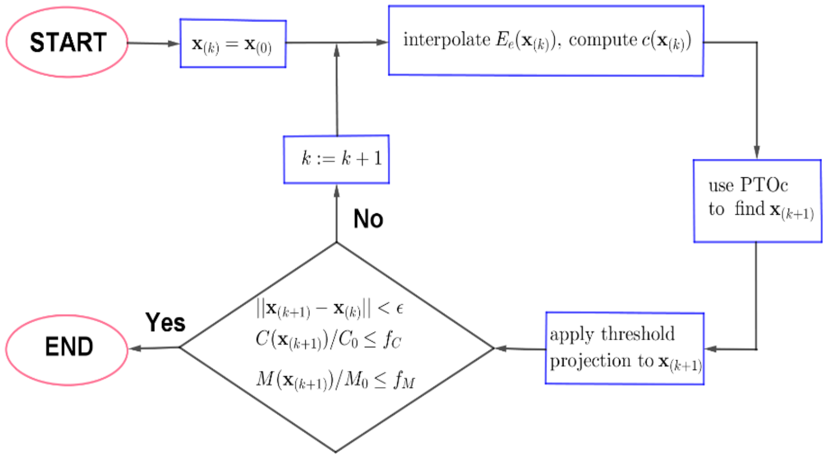


Figure 2. Flowchart of PTO for multi-material problem with mass and cost constraints.

3. NUMERICAL EXAMPLES

In this section, some numerical examples are presented. First, we examine the bridge model with normalized parameters. Then, we investigate a L bracket model related to real materials.

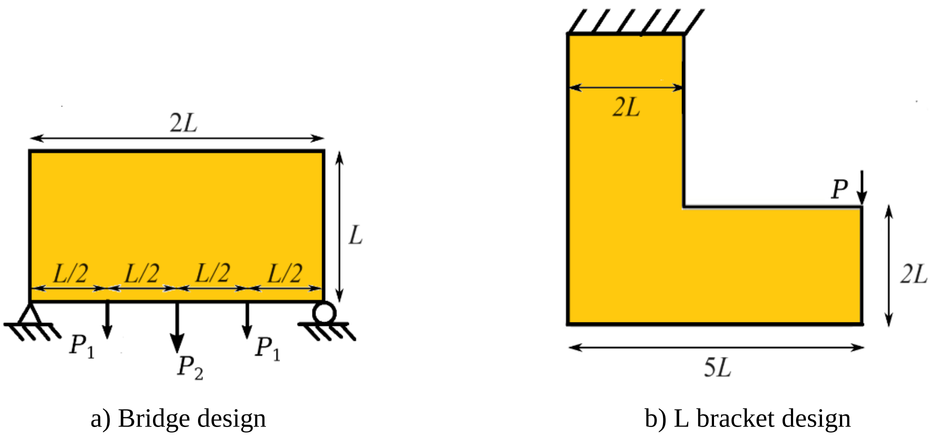


Figure 3. Geometries and boundary conditions of structure models.

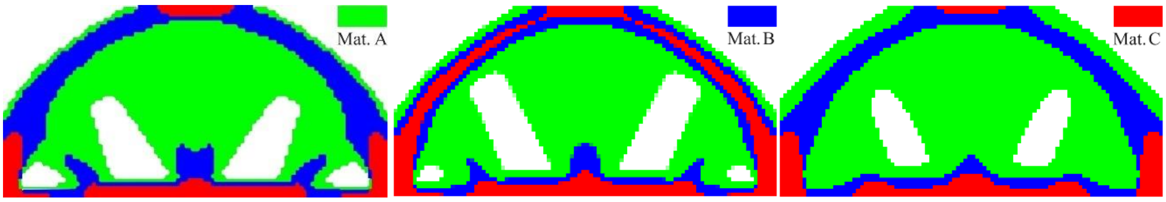
3.1. Four-phase bridge design

The minimum compliance topology optimization of a bridge structure is used to verify the proposed approach. The structure, boundary conditions and external forces are shown in Figure 3-a. All problems are treated as dimensionless. The external forces are $P_1 = 1$ and $P_2 = 2$. The minimum elastic modulus is $E_{\min} = 10^{-9}$ for the void phase, Poisson's ratio is $v = 0.3$, the domain size $L = 50$. The properties of candidate materials are listed in Table 1.

Table 1. Four-material properties.

	Void	Material A	Material B	Material C
ρ	0	0.4	0.7	1.0
E	$E_{\min}$	0.2	0.6	1.0
C	0	0.5	0.8	1.0
Color	White	Green	Blue	Red

Using the finite element method, the model employed a grid of 5000 quadrilateral elements, yielding 101 nodes horizontally and 51 nodes vertically. The problem was solved using an initial design variable $\mathbf{x}_{(0)} = 0.5$, subject to the constraints $f_M = 0.4$ and $f_C = 0.3$. Figure 4 provides a visual comparison between our solution and ones achieved by Zou [3]. Zou employs a gradient-based optimization approach using an SIMP interpolation, with penalty coefficients set to $p = 4$ and a filter radius of $r = 6$. In case of $r = 4$, our results closely align with Zou's findings concerning the distribution of void material ($\rho = 0$). Conversely, in case of $r = 6$, our results closely resemble Zou's solution for the distribution of material C ($\rho = 1$). Detailed comparisons between Zou's solution and ours, considering various filter radius values, are presented in Table 2. Notably, in all instances, our objective function yields lower values than Zou's results.



a) Zou's method with $r = 6$ b) Our method with $r = 4$ c) Our method with $r = 6$

Figure 4. Optimized topologies of bridge design in case of the comparison.

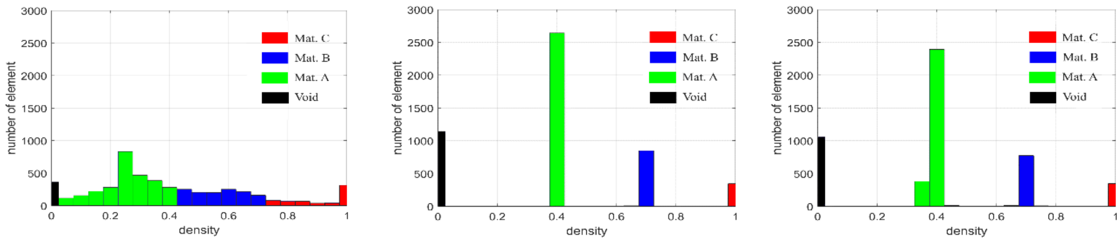
Table 2. Comparison of the optimal solution with respect to filter radius.

Method	Iteration	Compliance	Mass	Cost
Zou's method with $r = 6$	123	232.7	0.4	0.277
Our method with $r = 3$	51	203.5	0.4	0.297
Our method with $r = 4$	51	209.9	0.4	0.280
Our method with $r = 5$	51	215.2	0.4	0.265
Our method with $r = 6$	51	217.2	0.4	0.257

Our optimal solution with $r = 6$ provides results that are more conducive to the design process. Hence, all subsequent studies were conducted with a filter radius of $r = 6$.

3.1.1 Discussion on the projection parameter

Considering the problem with varying values of the projection parameter to understand its impact on the optimization results. The results are presented in Table 3. The notations M1, M2, M3 represent the measure of non-discreteness [10] for the intervals, which are defined by the densities of candidate materials. The smaller the measure of non-discreteness, the more the density values are concentrated around the values of the density candidate materials. We can see that as parameter b increases with the current iteration k in the PTO algorithm, the measure of non-discreteness becomes smaller (refer to Figures 5-b and 5-c). Conversely, when b is a constant, the density distribution is poor (see Figure 5-a). The following studies in this paper aim to achieve a low measure of non-discreteness.



a) $b=0$ b) $b=k/20$ c) $b=2+k/40$

Figure 5. Histogram of optimal density in case of various projection parameters.

Table 3. Comparison of the optimal solution with respect to projection parameter.

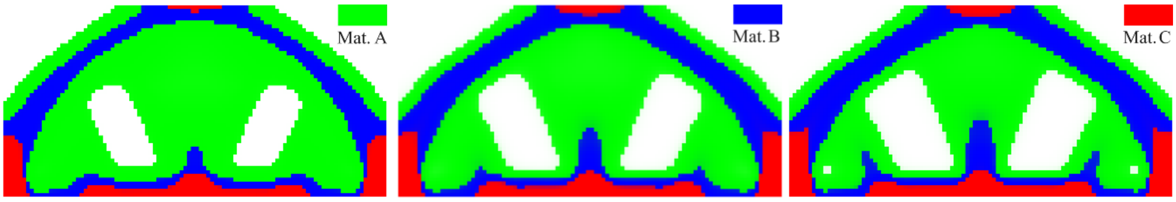
Projection parameter	Iteration	Compliance	Mass	Cost	M1	M2	M3
$b=0$	51	217.2	0.4	0.257	0.662	0.667	0.350
$b=2$	109	222.5	0.4	0.272	0.313	0.354	0.190
$b=4$	123	241.7	0.4	0.257	0.103	0.121	0.055
$b=k/20$	151	231.8	0.4	0.270	0.021	0.034	0.020
$b=k/40$	53	219.6	0.4	0.262	0.490	0.506	0.269
$b=2+k/40$	99	232.1	0.4	0.265	0.098	0.095	0.043

3.1.2 Influence of elastic modulus interpolation

In this study, the projection parameter is chosen to be in the linear form of $b=k/20$. The distribution of the elastic modulus is varied from $a=0$ (threshold interpolation) to $a=1$ (power interpolation). The results are presented in Figure 6 and summarized in Table 4. As a gradually increases, the computational cost decreases, reflected in the reduced number of iterations. The objective function value decreases slightly, but the measure of non-discreteness increases.

Table 4. Comparison of the optimal solution with respect to hybrid coefficient.

Hybrid coefficient	Iteration	Compliance	Mass	Cost	M1	M2	M3
$a=0$	151	231.8	0.4	0.270	0.021	0.034	0.020
$a=0.33$	109	228.7	0.4	0.275	0.052	0.075	0.033
$a=0.66$	81	226.6	0.4	0.283	0.078	0.111	0.069
$a=1$	86	224.8	0.4	0.289	0.059	0.1044	0.048



a) $a = 0$ b) $a = 0.66$ c) $a = 1$

Figure 6. Optimized solution in case of various hybrid coefficients.

3.1.3 Influence of initialization

In this study, the parameters are set as $b = k/25$ and $a = 0.5$. We investigate how the solution varies with initial values of $\mathbf{x}_{(0)}$, ranging from 0.2 to 0.6. The results are presented in Table 5. The cases with $\mathbf{x}_{(0)}$ = 0.3, 0.4 and 0.5 yield very favorable results, all with a small number of iterations. The case with $\mathbf{x}_{(0)}$ = 0.4 achieves the smallest objective function value, but it also exhibits a relatively high measure of non-discreteness. Conversely, the cases with $\mathbf{x}_{(0)}$ = 0.3 and $\mathbf{x}_{(0)}$ = 0.5 yield slightly higher objective function values but show very low values for the measure of non-discreteness.

Table 5. Comparison of the optimal solution with respect to initialization.

Initialization	Iteration	Compliance	Mass	Cost	M1	M2	M3
$\mathbf{x}_{(0)} = 0.2$	159	254.8	0.354	0.255	0.009	0.023	0.007
$\mathbf{x}_{(0)} = 0.3$	80	236.6	0.388	0.273	0.046	0.072	0.037
$\mathbf{x}_{(0)} = 0.4$	60	229.2	0.4	0.276	0.105	0.116	0.056
$\mathbf{x}_{(0)} = 0.5$	90	233.5	0.4	0.276	0.064	0.066	0.026
$\mathbf{x}_{(0)} = 0.6$	153	255.8	0.398	0.243	0.018	0.009	0.006

3.2. Steel and aluminum L bracket design

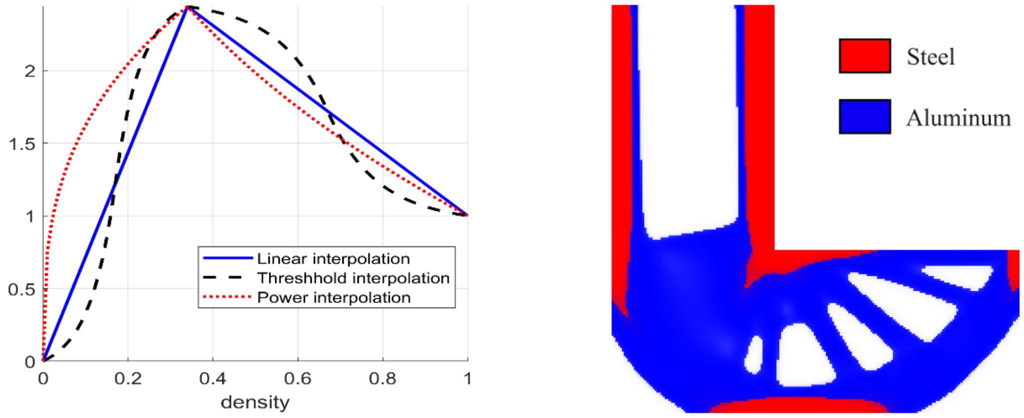
The topology optimization problem for steel and aluminum L bracket is studied. The structure, boundary conditions and external forces are shown in Figure 3-b. The properties of candidate materials and their normalized values are listed in Table 6. The external force $P = 1$ and the domain size $L = 40$. The finite element method is applied with a mesh of 25600 four-node elements. The parameters are set as follows: $r = 6$, $a = 0.5, b = 1 + k/60$, $f_M = 0.35$, $f_C = 0.55$.

This research contributes to understanding the impact of cost function distributions on the solutions of the optimization problem. Multi-material cost distributions are examined in various scenarios, including linear, threshold, and power interpolations, as depicted in Figure 7-a. It's important to note that in this problem, the cost of the lightweight material (aluminum) is significantly higher than that of the heavy material (steel). Cost constraints are carefully selected to play a decisive role in determining the optimal solution. An illustration of the solution corresponding to the linear cost distribution is provided in Figure 7-b. The results are presented in Table 7. In the case of power interpolation, the total cost is very low. This can be explained by the fact that when the density is concentrated around the densities of candidate materials, the cost associated with power interpolation is minimized compared to others, especially when the material volume is predominantly composed of steel. This aligns with the real-world scenario where steel is both cost-effective and sturdy. However, aluminum also has its strengths with unique mechanical properties.

Table 6. Steel and aluminum material properties.

Material properties			
	Void	Aluminum	Steel
$\rho (kg/m^3)$	0	2700	7850
E (GPa)	0	69	210
C (€/kg)	0	1.71	0.70
Color	White	Blue	Red

Normalized properties			
	Void	Aluminum	Steel
ρ	0	0.34	1.0
E	E_{min}	0.33	1.0
C	0	2.44	1.0
Color	White	Blue	Red



a) Cost in various interpolations b) The solution with linearinterpolation

Figure 7. Optimized topology of steel and aluminum L bracket design.

Table 7. Comparison of the optimal solution with respect to hybrid coefficient.

Interpolation	Iteration	Compliance	Mass	Cost	M1	M2
Linear	155	227.2	0.327	0.550	0.080	0.079
Threshold	142	229.9	0.323	0.550	0.089	0.086
Power	124	219.7	0.342	0.232	0.099	0.106

4. CONCLUSIONS

This paper presents a hybrid threshold interpolation method for the elastic modulus to address multi-material topology optimization problems with mass and cost constraints. The minimum compliance topology optimization problem is tackled using the PTO method. Density variables undergo a projection process to align with candidate material densities.

In cases where threshold projection is not applied, the PTO method produces superior results, achieving lower objective function values than the ordered SIMP method. Moreover, the number of iterations done by the PTO method is substantially fewer than that of SIMP. However, the density distribution in both methods is poor. To address this, threshold projection is employed to transform the density, concentrating density variables on candidate material values. Numerical examples demonstrate that selecting the projection parameter b as a linear function is a suitable choice.

When combining two distributions of the elastic modulus, it is observed that as the elastic modulus tends toward a power function, the objective function decreases, but the measure of non-discreteness increases. Conversely, when the elastic modulus tends toward a threshold function, the opposite effect is observed. To resolve this issue, a combination with hybrid coefficient a ranging between 0 and 1 can be chosen as a solution depending on specific requirements. The initial values of design variables also impact the optimization results. It is advisable to select initial design variables such that the initial mass slightly exceeds the upper mass constraint.

By synthesizing results from various studies on the influence of parameters such as filter radius, projection parameter, hybrid coefficient, and initial design variables, we can determine an appropriate model to effectively address each specific optimization problem.

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***Declaration of competing interest*.** The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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