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HELMHOLTZ AND LIÉNARD-TYPE OSCILLATORS FROM NON-STANDARD LAGRANGIANS

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Abstract. This study investigates the nonlinear second-order differential equation $\ddot{x} + f(x)\dot{x}^n + g(x)w(x) = 0, n = 1, 2, 3, \ldots$ and $\ddot{x} + g(t)\dot{x}^m + k(t)W(x) = 0, m = 1, 2, 3, \ldots$ which model a class of dynamical systems characterized by velocity-dependent nonlinearities and nontrivial oscillatory behavior. By employing the method of nonstandard Lagrangians, we derive a systematic variational framework for such equations, enabling the identification of a number of properties that are not accessible through traditional Lagrangian formulations. The resulting first integrals and implicit solutions provide insight into the complex dynamics of the system, including amplitude-dependent oscillations and stability properties. This approach highlights the relevance of nonstandard Lagrangians in capturing the behavior of nonlinear, dissipative, or nonconservative systems, offering both theoretical and practical tools for the analysis of advanced mechanical, physical, and engineering models. Our approach may be used also, after a suitable change of coordinate, to describe complex biological systems such as the Susceptible–Infectious–Recovered model. An analogy with a position-dependent mass system is also addressed.

Keywords: non-standard Lagrangians, nonlinear oscillations.

1. INTRODUCTION

The huge development of computer-aided techniques used for solving nonlinear dynamical systems has naturally pushed many researchers in different fields of science to extend diverse non-standard calculus of variations in many dissimilar contexts. During the last decades, there has been more attention paid to studying nonlinear oscillators with quadratic damping (NOQD) because of the enormous series of applications in applied sciences and mainly in hydrological drag and aerodynamics problems (Madison, 2012; Yamamoto & Nath, 1976) among others (Fay, 2012; Smith, 2012). Quadratic damping arises in general in fluid flows subject to high velocity. In general, the drag force is the sum of linear and quadratic terms. The linear term dominates the dynamics for slow fluid flows, whereas for high flow velocity the quadratic term is dominant. This behavior is useful to explore isenthalpic oscillations in a saturated two-phase fluid as done in Madison (2012) and to study the oscillatory flows around a cylinder as done in Yamamoto and Nath (1976). To model the phenomenon of damping, in particular at higher velocities one needs to introduce by hand a quadratic nonlinearity in the dynamical equation of motion mainly in the form $-sign(\dot{x})\dot{x}^2$, $\dot{x} = \mathrm{d}x/\mathrm{d}t$ being the velocity of the body (Pandey et al., 2016). Usually, the equation of motion with quadratic damping is tricky to solve, and most of these equations require numerical approaches. In general, nonlinear dynamical sciences arise in several fields of science and engineering, e.g., in mechanics, astrophysics, hydrodynamics, and so on (Epstein & Pojman, 1998; Miwadinou, Monwanou, Hinvi, & Orou, 2018; Miwadinou, Monwanou, Koukpemedji, et al., 2018; Nayfey & Mook, 1979; Olabodé et al., 2019; Poston & Stewart, 1978). The Helmholtz oscillator is one typical example of a nonlinear oscillator that arises in acoustics and naval engineering (Spyrou et al., 2002; Thompson, 1997) and is used to analyze the dynamics of the Lotka–Volterra system (Fangnon et al., 2020), where a generalized Helmholtz oscillator containing linear, impure quadratic, and pure quadratic damping coefficients was introduced. One of the original features of the nonlinear damping model introduced in Fangnon et al. (2020) is the new model of the Helmholtz equation obtained by reducing the number of variables of the prey-predator Lotka–Volterra system. The main aim of the present work is to prove that the generalized Helmholtz equation, or NOQD, may be obtained from the fundamental problem of the calculus of variations through different types of Lagrangian functionals motivated from geometrical setup. The Lagrangian function serves as a central element in describing dynamical systems and offers an amalgamated frame for addressing a large band of complex physical problems. Lagrangian functions are required to describe real-world dynamical problems, mainly when using the variational approach.

The first type of Lagrangian is standard and has the mathematical form $L_1 = e^{\beta x} L$ where β is a real constant and L is the conventional Lagrangian. In geometrical field theories, the factor $e^{\beta x}$ is identified to a dilaton field. The second type has the mathematical non-standard Lagrangian (NSL) structure $L_2 = e^{\beta x} L^{1+\varepsilon}$, ε is another real constant. The third type is $L_3 = e^{\beta x} e^{\varepsilon L}$ which extends the exponential NSL. We would like also to prove that NOQD may be also obtained from these special forms of Lagrangians and also from their higher-order derivative structures, i.e., higher-order non-standard Lagrangians (HONSL). In general, higher-order Lagrangian equations containing the higher-order time derivative of generalized coordinates play a crucial role in different branches of applied sciences and theoretical physics (Gràcia et al., 1991; Hong-Xia et al., 2008; Mei et al., 1991). Besides, recent advances in the geometric formulation of the higher-derivative Hamilton and Lagrangian dynamics show that the latter can give a precise and natural formulation of many aspects of field theories (Román-Roy, 2009; Skinner, 1983; Vitagliano, 2010). Such a formulation is familiar and is based on a generalization to higher derivative Lagrangian field theory of the mixed Lagrangian–Hamiltonian formalism, i.e., including higher-order derivatives of the generalized coordinates with respect to time (Andrianov et al., 1996; Bender & Mannheim, 2008; Borges et al., 2019; Crampin et al., 2010; T. Kamalov, 2013; T. F. Kamalov, 2009, 2020; Leon & Rodrigues, 1985; Simon, 1990a, 1990b; Skinner & Rusk, 1983a, 1983b; Suykens, 2009). Higher-order derivatives in classical mechanics date back to 1848 since Ostrogradsky and Jacobi introduced the basis of "generalized classical mechanics" (Crampin et al., 2010). The topic was chiefly explored in theoretical physics and applied mathematics ranging from electromagnetic theory with second-order derivatives to field theories and modern geometric methods (Aldaya & Azcarraga, 1992; Simon, 1990a; Vitagliano, 2011; Vitagliano, 2010; Yong-Fen et al., 2002). From the other side, the theory of NSL, which is characterized by "deformed kinetic energy and potential energy terms," has recently gained particular awareness due to its motivating implications in applied mathematics and theoretical physics, mainly dissipative dynamical systems and the theory of nonlinear differential equations (Cariñena & Núñez, 2016a, 2016b; Cariñena & Guha, 2019; Cariñena et al., 2005; Cieśliński & Nikiciuk, 2010; Davachi & Musielak, 2019; El-Nabulsi, 2012, 2013a, 2013b, 2013c, 2014, 2015a, 2020b; El-Nabulsi et al., 2013; El-Nabulsi, 2013d; El-Nabulsi, 2015b; Jiang et al., 2019; Musielak, 2008, 2009; Musielak et al., 2008, 2020a; Musielak et al., 2020b; Saha & Talukdar, 2014; Song & Zhang, 2018; Zhang & Wang, 2019; Zhang & Zhou, 2016; Zhou & Zhang, 2016). NSL are just the generating functions for the basic equations, as pointed out by Arnold (1978). As proved by Musielak et al. (2020a) and Musielak et al. (2020b), NSL should be regarded as new generating functionals for dynamical equations. A large number of conventional equations of motion may be derived from both standard and NSL if supplementary constraints connected to the Lie group were used, which leads to a novel phenomenon in the calculus of variations.

The paper is organized as follows: In Section 2, we introduce three types of Lagrangians, we set up their corresponding Euler–Lagrange equations, and we discuss several illustrations; in Section 3, we discuss their associated higher-order derivative Lagrangians; a discussion is done in Section 4, and conclusions are given in Section 5.

2. STANDARD AND NON-STANDARD LAGRANGIANS

To start with the first standard Lagrangian $L_1 = e^{\beta x}L$, we consider a one-dimensional dynamical system. Let $x \in C^1[a,b]$, $L(\dot{x},x,t)$ be a smooth extended Lagrangian function where $(\dot{x},x,t) \rightarrow L_1(\dot{x},x,t) \equiv L_1$ (for convenience) is assumed to be a C^2 function with respect to its arguments and satisfying fixed boundary conditions $x(a) = x_a$ and $x(b) = x_b$ with corresponding action functional defined by: $S = \int_a^b L_1(\dot{x},x,t)\mathrm{d}t = \int_a^b e^{\beta x}L\mathrm{d}t$. It should be stressed that the idea of introducing an exponential global factor in a Lagrangian was addressed in the literature by Caldirola (1941) and Kanai (1948). In their approach, an oscillator of mass m and subject to a damping term $\eta \propto m^{-1}$ and a frequency ω_0 is characterized by the Lagrangian $L = \frac{1}{2}e^{\eta t}(\dot{x}^2 - \omega_0^2 x^2)$. The conventional lemma of the calculus of variations yields the following 2^{nd}-order differential equation $\ddot{x} + \eta\dot{x} + \omega_0^2 x = 0$ satisfying the Euler–Lagrange equation and which is characterized by a linear damping term. A comparable approach has been introduced by Bateman (1931) who introduced the Lagrangian $L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy$ (m is the mass of the body, γ is the damping constant, and k is the spring constant), The Euler–Lagrange equation gives $m\ddot{x} + \gamma\dot{x} + kx = 0$ for the damped oscillator, and $m\ddot{y} - \gamma\dot{y} + ky = 0$ for the amplified oscillator (emergence of a negative friction) (Vestal & Musielak, 2021). The transformation $(x,y,\gamma) \rightarrow (y,x,-\gamma)$ relates the two equations. The resulting Hamiltonian gives irrelevant solutions that must be eliminated. For the case of damped oscillators, the Lagrangian is time-dependent. The construction of time-independent Lagrangian functions for dissipative systems has been discussed largely in the literature, and one typical example is the work of Bateman. In our approach, the exponential global factor is position-dependent and not time-dependent. In Bateman and Caldirola–Kanai approaches, quadratic nonlinearity in the dynamical equation of motion is not obtained. In general, the Lagrangian $L_1 = e^{\beta x}L$ has not been considered in the literature to the best of our knowledge. Therefore, it will be interesting to study the effects of

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \beta L - \beta\dot{x}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1}$$

The proof of Eq. (1) is straightforward and is a consequence of the principle of the calculus of variations (Cline, 2018). On the other hand, for $L_2 = e^{\beta x}L^{1+\varepsilon}$ the following Euler–Lagrange equation holds:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \beta\dot{x}\frac{\partial L}{\partial \dot{x}} - \frac{\varepsilon}{L}\frac{\partial L}{\partial \dot{x}}\left(\dot{x}\frac{\partial L}{\partial x} + \ddot{x}\frac{\partial L}{\partial \dot{x}}\right) + \frac{\beta}{1+\varepsilon}L = 0. \tag{2}$$

Eq. (2) is reduced to Eq. (1) for $\varepsilon = 0$. In the following subsections, a number of illustrations will be presented for each type of Lagrangians:

2.1. The case of the standard Lagrangian $L_1 = e^{\beta x}L$

We start by the following Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - m^2 x, \tag{3}$$

where m is a real constant. Eq. (1) gives the following NOQD:

$$\ddot{x} + \frac{\beta}{2}\dot{x}^2 + \beta m^2 x + m^2 = 0. \tag{4}$$

Eq. (4) corresponds in fact to an oscillator with strong quadratic damping force (Cveticanin, 2009) (Helmholtz equation) and the solution is given in terms of the Jacobi elliptic function cn by:

$$x(t) = -\frac{1}{\beta} + A + B\mathrm{cn}^2(\pm\omega t + C, K^2), \tag{5}$$

where cn is the Jacobi elliptic function, ω is the frequency, K is the modulus of the function and A, B, C are real constants. In fact, deriving Eq. (5) with respect to time and after replacing into Eq. (4) gives:

$$\omega^2 = \frac{3 + 2\beta B}{2\left(\beta^2 B^2 + \frac{4}{3}\beta B + 6\right)}, \tag{6}$$

$$K^2 = \frac{\beta B}{2\beta B + 3}, \tag{7}$$

$$A + m^2 = \frac{-3B + \beta B^2}{6 + 2\beta B + \beta^2 B^2}, \tag{8}$$

whereas the constants B and C are obtained from the initial conditions: $x(t_0) = q(0)$ and $\dot{x}(t = t_0) = \dot{x}(0)$ where

$$\dot{x} = -2B\omega\mathrm{cn}\left(\pm\omega t + C, K^2\right)\mathrm{sn}\left(\pm\omega t + C, K^2\right)\mathrm{dn}\left(\pm\omega t + C, K^2\right). \tag{9}$$

Here sn and dn are the Jacobi elliptic functions (Byrd & Friedman, 1954). It is obvious from Eqs. (6) and (7) that $\omega^2 > 0$ and $K^2 > 0$ if $B < 0$ and $\beta < 0$ or $B > 0$ and $\beta > 0$. We plot in Fig. 1 the variations of Eq. (5) after fixing the parameters for suitable values. The following MATLAB program 1 is used:

```
clear all
clc
A = 1;
B = 1;
C = 0;
K = sqrt(5);
beta = -15/9;
omega = [1 5];
t = linspace(0, 3, 500);
m = K^2;
fig1 = figure('Color', 'w');
hold on;
for i = 1:length(omega)
    u = omega(i) .* t + C;
    x = (-1/beta) + A + B .* jacobiCN(u, m).^2;
    plot(t, x, 'LineWidth', 2, 'DisplayName', ['\omega = ' num2str(omega
        (i))]);
end
xlabel('t');
ylabel('x(t)');
title('x(t) depending on \omega (positive frequency)');
legend('Location', 'best');
hold off;
exportgraphics(fig1, 'signe_de_omega_positive.png', 'BackgroundColor', '
    white');

```

```
fig2 = figure('Color', 'w');
hold on;
for i = 1:length(omega)
    v = -omega(i) .* t + C;
    y = (-1/beta) + A + B .* jacobiCN(v, m).^2;
    plot(t, y, 'LineWidth', 2, 'DisplayName', ['\omega␣=␣' num2str(omega
        (i))]);
end
xlabel('t');
ylabel('x(t)');
title('x(t)␣depending␣on␣-\omega␣(negative␣frequency)');
legend('Location', 'best');
hold off;
exportgraphics(fig2, 'signe_de_omega_negative.png', 'BackgroundColor', '
    white');
```

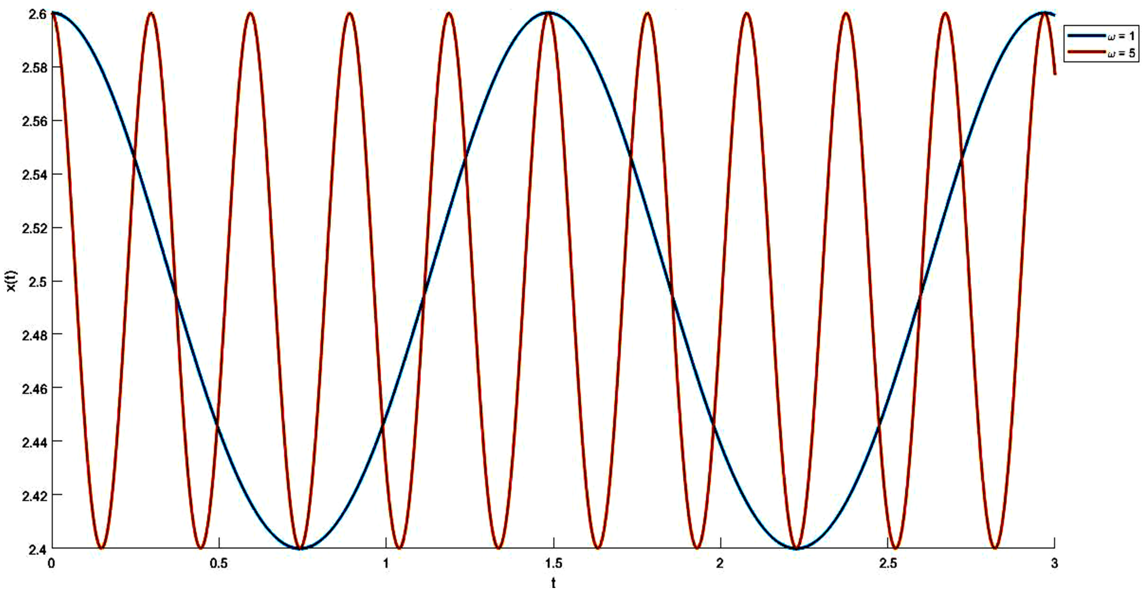

Fig. 1. Variations of Eq. (5) for two different ω

Surprisingly, we have obtained a quadratic damping oscillator starting from a linear potential and not by the conventional quadratic potential. In fact, a linear potential is widely explored in quantum mechanics, e.g., quantum motion of a particle in a gravitational field (Berberan-Santos et al., 2005). For small K, we may use the following rules (Milne-Thomson, 1950):

$$\mathrm{sn}\left(\omega t, K^2\right) = \sin\left(\omega t\right) - \frac{1}{4}K^2 \cos\left(\omega t\right)\left(\omega t - \frac{1}{2}\sin\left(2\left(\omega t\right)\right)\right), \tag{10}$$

$$\mathrm{cn}\left(\omega t, K^2\right) = \cos\left(\omega t\right) + \frac{1}{4}K^2 \sin\left(\omega t\right)\left(\omega t - \frac{1}{2}\sin\left(2\left(\omega t\right)\right)\right), \tag{11}$$

$$\mathrm{dn}\left(\omega t, K^2\right) = 1 - \frac{1}{2}K^2 \sin^2\left(\omega t\right), \tag{12}$$

and the graphical solution is represented accordingly by Fig. 2.

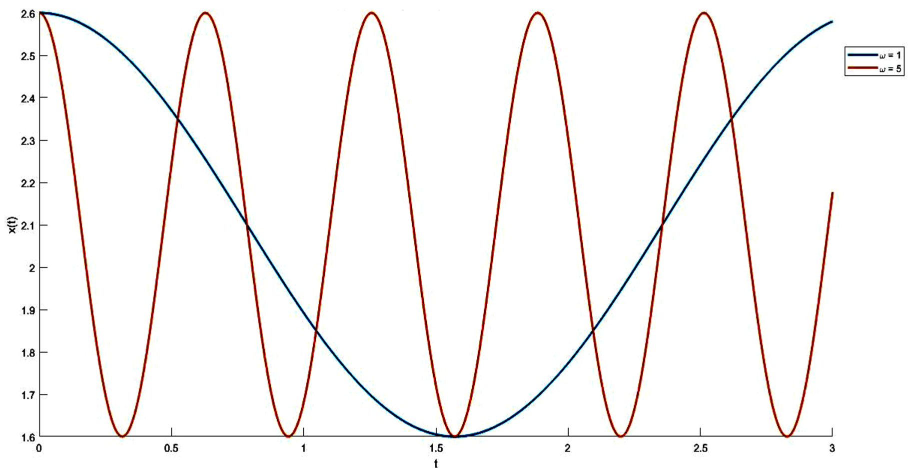

Fig. 2. Approximate solutions by the design scheme for different values of the frequency. We observe that the frequencies of oscillations are slower than those illustrated in Fig. 1, which is expected due to the mathematical nature of the Jacobi elliptic functions

We plot in Fig. 3, the variations of the phase space for three different values of the frequency based on MATLAB program 2:

```
clear all
clc
A = 1;
B = 1;
C = 0;
K = sqrt(5);
beta = -15/9;
omega = [1 5 10];
t = linspace(0, 5, 150);
m = K^2;
fig1 = figure('Color', 'w');
hold on;
for i = 1:length(omega)
    u = omega(i) .* t + C;
    cn = jacobiCN(u, m);
    sn = jacobiSN(u, m);
    dn = jacobiDN(u, m);
    x = (-1/beta) + A + B .* cn.^2;
    dxdt = -2 * B * omega(i)^2 .* cn .* sn .* dn;
    plot(x, dxdt, 'LineWidth', 1, 'DisplayName', ['\omega␣=␣' num2str(
        omega(i))]);
end
xlabel('x');
ylabel('dx/dt');
title('Phase␣Portrait:␣dx/dt␣vs.␣x␣for␣positive␣\omega');
legend('Location', 'best');
hold off;
exportgraphics(fig1, 'signe_de_omega_positive.png', 'BackgroundColor', '
    white');

fig2 = figure('Color', 'w');
hold on;
```

```
for i = 1:length(omega)
    u = -omega(i) .* t + C;
    cn = jacobiCN(u, m);
    sn = jacobiSN(u, m);
    dn = jacobiDN(u, m);
    x = (-1/beta) + A + B .* cn.^2;
    dxdt = -2 * B * omega(i)^2 .* cn .* sn .* dn;
    plot(x, dxdt, 'LineWidth', 1, 'DisplayName', ['-\omega = ' num2str(-omega(i))]);
end
xlabel('x');
ylabel('dx/dt');
title('Phase Portrait: dx/dt vs. x for negative \omega');
legend('Location', 'best');
hold off;
exportgraphics(fig2, 'signe_de_omega_negative.png', 'BackgroundColor', 'white');
```

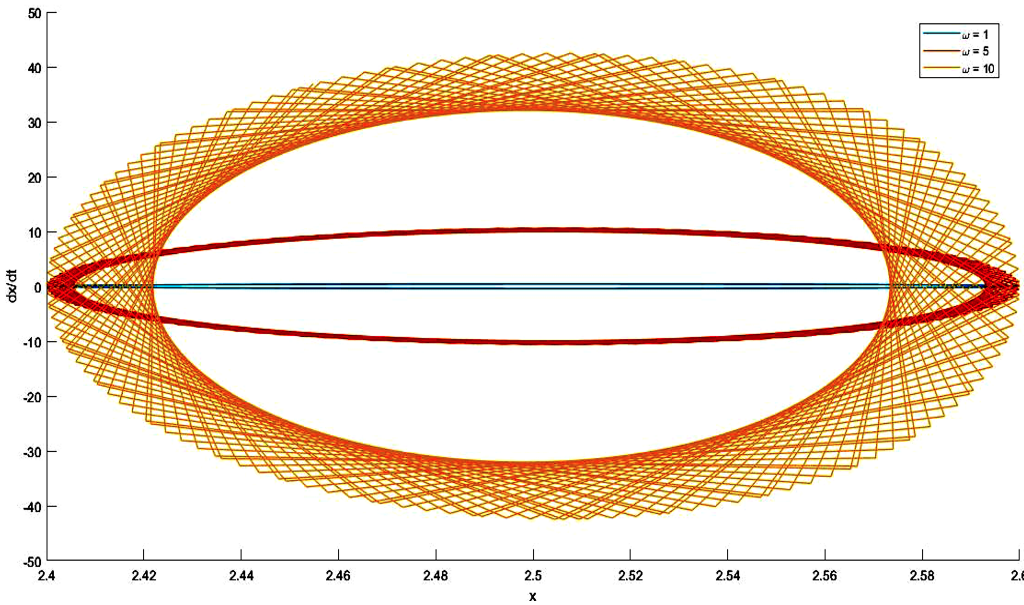

Fig. 3. This figure illustrates the phase plane between velocity and displacement of the system for different frequencies. Each point in the phase space represents a possible state of the system. The orbits are closed and have constant energy. For large amplitudes, the tori broaden and begin to extend beyond and finally merge to form a single identical enlarged torus. At this point, the oscillator undergoes quasiperiodic motions

For small K, the following phase portrait in Fig. 4 holds.

As a 2^{nd} illustration, we consider the following Lagrangians:

$$L = \tfrac{1}{2}\dot{x}^2 - m^2(x + \dot{x}), \tag{13}$$

$$L = \tfrac{1}{2}\dot{x}^2 - \alpha x\dot{x} - m^2 x, \tag{14}$$

where α is a real parameter. Both Lagrangians result in the following NOQD:

$$\ddot{x} + \frac{1}{2}\beta\dot{x}^2 + \beta m^2 x + m^2 = 0. \tag{15}$$

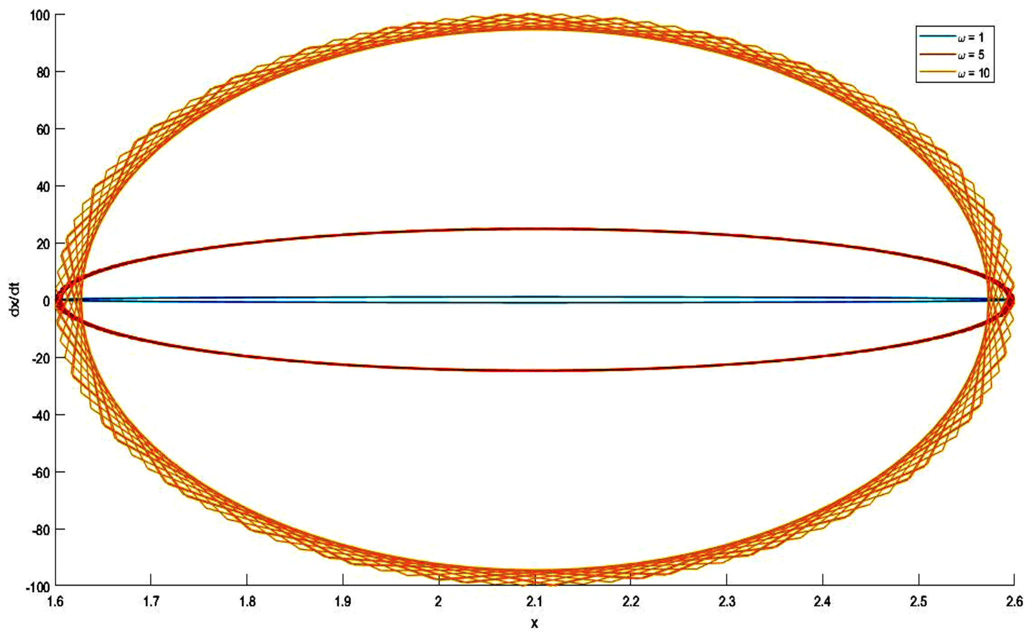

Fig. 4. This figure illustrates the phase plane between velocity and displacement of the system for different frequencies for small K. As we increase the amplitude, the torus does not largely broaden, yet it begins to extend beyond to form a single identical enlarged torus

This equation is comparable to Eq. (4), and the solution is given by Eq. (5). Eq. (4) or (15) belongs in general to the family of Helmholtz oscillators used in control mechanisms (Sanjuán, 1999). The Lagrangians (13) and (14) are gauge equivalent to the Lagrangian (3), therefore leading to similar results; the only point is that when multiplied by the function $e^{\beta x}$ the added gauge term is not a gauge term anymore.

Remark 1. *It is notable that we have considered a standard, or mechanical type Lagrangian L, which is defined by a Lagrangian defined by a conformally flat Riemann structure and a basic function (see (Benenti, 2005; Cariñena et al., 2014; Crampin & Sarlet, 2001) and references therein), i.e., we only consider as L one defined by a multiple of a Euclidean metric (the factor is the constant mass). Then, L_1 is an interesting example of natural Lagrangians, in the generalized sense, for which the Riemann structure is conformally flat, i.e., a multiple by a function, which plays the role of a position-dependent mass $m(x)$ of a Euclidean metric. These systems are called position-dependent mass systems and have received a lot of attention during the last years (Cruz y Cruz et al., 2008; Cruz y Cruz & Rosas-Ortiz, 2009; Dong et al., 2007; El-Nabulsi, 2020a, 2020c; Eshghi et al., 2018; Mustafa, 2015; Yu et al., 2004). To prove this, one can consider the Lagrangian $L = \frac{1}{2}m(x)\dot{x}^2 - V(x)$. It is easy to verify that its associated equation of motion is given by $\ddot{x} + \frac{1}{2}\frac{m'(x)}{m(x)}\dot{x}^2 + \frac{1}{m}V'(x) = 0$. By letting $m(x) = e^{\beta x}$, we obtain the quadratic Liénard type equation $\ddot{x} + \frac{1}{2}\beta\dot{x}^2 + \frac{1}{m}V'(x) = 0$ which is well-known in the literature (Gubbiotti & Nucci, 2015; Tiwari et al., 2013). Therefore a system with a quadratic in velocity damping $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ corresponds to the motion of a position-dependent mass system for a mass function such that $m(x) = \exp(2\int^x f(\zeta)\mathrm{d}\zeta)$ and a potential function $V(x) = \int^x g(\zeta)\mathrm{d}\zeta$. For $V(x) = m^2 x e^{\beta x}$, we obtain equation (4).*

2.2. The case of the standard Lagrangian $L_2 = e^{\beta x}L^{1+\varepsilon}$

We start by considering the NSL:

$$L = \dot{x} + mx, \tag{16}$$

where m is a real parameter and $i = \sqrt{-1} \in \mathbb{C}$. It is easy to verify using Eq. (2) that the resulting equation of motion takes the subsequent form:

$$\varepsilon\ddot{x} - \underbrace{(1-\varepsilon)m\dot{x}}_{\text{linear damping}} - \underbrace{m\beta\frac{1-\varepsilon}{1+\varepsilon}x\dot{x}}_{\text{hybrid damping}} + \underbrace{\frac{\beta\varepsilon}{1+\varepsilon}\dot{x}^2}_{\text{pure quadratic damping}} \underbrace{-\frac{\beta}{1+\varepsilon}m^2x^2 - m^2x}_{\text{terms generated from a quartic potential}} = 0. \quad (17)$$

This equation belongs to the modified Helmholtz equation introduced in Fangnon et al. (2020), or the so-called Liénard-type second-order differential equation which takes the form $\ddot{x} + f(x)\dot{x}^2 + g(x)\dot{x} + w(x) = 0$. In particular for $\varepsilon = 1$ and $m \to im$ with $i = \sqrt{-1} \in \mathbb{C}$, Eq. (17) is reduced to:

$$\ddot{x} + \frac{\beta}{2}\dot{x}^2 + m^2x + \frac{\beta}{2}m^2x^2 = 0, \quad (18)$$

where the solution is given by in general $x = a\cos(\Omega t + \varphi)$ where a is the amplitude, Ω is the frequency and φ is the phase of oscillations. These parameters are subject to several constraints discussed in detail in Fangnon et al. (2020). It was observed the solution exhibits phenomena of resonance, hysteresis, and chaos. Similar dynamics were detected also in Olabodé et al. (2019) in the analysis of nonlinear dynamics of chemical oscillations governed by a forced modified Van der Pol–Duffing oscillator subject to fluctuating hydrodynamic drag forces. In Hu (2006, 2007), Mickens (1995), Porwal and Vyas (2008), and Soliman and Thompson (1992), comparable nonlinear oscillators having quadratic and mixed-parity types of nonlinearities in stiffness were studied. It should be stressed that deriving the correct equation of motion from a complexified action was discussed largely in literature ranging from classical to quantum dynamics (El-Nabulsi, 2009, 2017; El-Nabulsi & Wu, 2012; Mandal & Mahajan, 2015; Sbitnev, 2009). Notably, these complexified Lagrangian systems are also characterized by complex Hamiltonians. One might ask what happens if solutions are not real-valued for physical observables. One naturally expects the energy to be complex, which may deform the trajectories of the particle, i.e., complex and chaotic trajectories as observed in quantum systems (Bender et al., 2007). However, the association of classical mechanics with quantum systems having continuous spectra requires a careful study.

A slight change in the NSL (16) in the form $L = x + m\dot{x} + a$ with a being a real parameter will result in the following generalized Helmholtz equation:

$$\begin{aligned} &\varepsilon m^2\ddot{x} - \underbrace{\left(1-\varepsilon-a\beta+\frac{2a\beta}{1+\varepsilon}\right)m\dot{x}}_{\text{linear damping}} - \underbrace{\left(\frac{2}{1+\varepsilon}-1\right)\beta m x\dot{x}}_{\text{hybrid damping}} \\ &\underbrace{-\beta m^2\left(\frac{1}{1+\varepsilon}-1\right)\dot{x}^2}_{\text{pure quadratic damping}} - \underbrace{\frac{\beta}{1+\varepsilon}x^2 - \left(\frac{2a\beta}{1+\varepsilon}+1\right)x - a + \frac{\beta a^2}{1+\varepsilon}}_{\text{terms generated from quartic potential}} = 0. \end{aligned} \quad (19)$$

where the solution is presented also in Fangnon et al. (2020) and Olabodé et al. (2019). The main difference between Eqs. (17) and (19) lies in the form of the quartic potential. In Eq. (14), the quartic potential is $V(x) = A_1x^4 + B_1x^2 + C_1$ whereas in (19) it takes the form $V(x) = A_2x^4 + B_2x^2 + C_2x + D_1$ where $A_1, B_1, \ldots$ are real parameters.

As a 2^{nd} illustration, we consider the NSL:

$$L = (\dot{x} - mx)e^{cx}, \quad (20)$$

where c is a real parameter. Eq. (20) gives the following generalized Helmholtz equation:

$$\begin{aligned}&\varepsilon\ddot{x}-\underbrace{\left(\frac{\beta}{1+\varepsilon}-\beta-\varepsilon c\right)\dot{x}^2}_{\text{pure quadratic damping}}-\underbrace{m\left(\beta-c-\frac{2\beta}{1+\varepsilon}+\varepsilon c\right)x\dot{x}}_{\text{hybrid damping}}\\&-\underbrace{m(\varepsilon-1)\dot{x}}_{\text{linear damping}}-\underbrace{m^2\left(\frac{\beta}{1+\varepsilon}+c\right)x^2-m^2x}_{\text{quartic potential}}=0.\end{aligned}\tag{21}$$

One more time, for specific values of the parameter, this equation may be reduced to an equation similar to Eq. (15) and its solution is also given by $x=a\cos(\Omega t+\varphi)$(Fangnon et al., 2020; Olabodé et al., 2019). These illustrations prove the effect of NSL for producing the modified or generalized Helmholtz equation suitable to describe nonlinear oscillations physics. Again, Eq. (21) may describe, for suitable choices of the parameters, a quartic oscillator.

Remark 2. *It is possible to eliminate the quadratic change in the Liénard-type differential second-order differential equation by introducing the change of variables* $\mathrm{d}y=e^{\int f(x)\mathrm{d}x}\mathrm{d}x$ *which converts* $\ddot{x}+f(x)\dot{x}^2+g(x)\dot{x}+w(x)=0$ *to* $\ddot{y}+g(y)\dot{y}+\tilde{w}(y)=0$ *where* $\tilde{w}(y)=e^{\int f(t)\mathrm{d}t}w(t)$ *(Paliathanasis & Duffy, 2025). In case* $f(x)=f_0$ *(a constant), then* $\mathrm{d}y=e^{f_0t}\mathrm{d}t$ *and* $\tilde{w}(y)=e^{f_0t}w(t)$*. These hold for Eqs. (17) and (19). For specific forms of* $g(y)$ *and* $\tilde{w}(y)$*, e.g.,* $g(y)=K$ *(a constant) and* $\tilde{w}(y)=y$*, the equation of a damped oscillator is obtained. Whereas, for* $g(y)=3y$ *and* $\tilde{w}(y)=y^3$*, the Painlevé–Ince equation is obtained (Llibre, 2023; Tshibase & Govinder, 2025). This equation is of particular interest due to its attractive properties (Halder et al., 2019; Mahomed & Leach, 1985). Hence, for specific factors, Eqs. (17) and (19) are maximally symmetric and could be linearized.*

Observe that, by selecting the NSL $L=\dot{x}+e^x$, we find the following dynamical equations:

$$\varepsilon\ddot{x}+\left(\beta-1+\varepsilon-\frac{2\beta}{1+\varepsilon}\right)\dot{x}e^x+\frac{\beta\varepsilon}{1+\varepsilon}\dot{x}^2-\left(\frac{\beta}{1+\varepsilon}+1\right)e^{2x}=0.\tag{22}$$

By letting $\mathrm{d}y=e^{\int f(t)\mathrm{d}t}\mathrm{d}t$ with $f(x)=f_0=\frac{\beta}{1+\varepsilon}$ and fixing $\varepsilon=\frac{2\beta}{1+\varepsilon}+1-\beta$, then Eq. (19) is reduced to:

$$\ddot{y}-\frac{1}{\varepsilon}(f_0+1)\,e^{(2+f_0)t}=0.\tag{23}$$

In particular, for $f_0=-1$, we obtain $\ddot{y}=0$ which corresponds to a free particle described by the Lagrangian $L=\frac{1}{2}y'^2$. By letting $\mathrm{d}y=e^{\int e^F\mathrm{d}T-\alpha T}\mathrm{d}T,\alpha\in\mathrm{R}$, equation $\ddot{y}=0$ is converted to $\ddot{T}+(\alpha-e^T)\dot{T}=0$ which is the same equation obtained in SIR model (the Susceptible–Infectious–Recovered model) describing the dynamics of an infectious disease within a population (Magal & Ruan, 2014) obtained after using by using point transformations (Paliathanasis & Duffy, 2025). We argue that our formalism may also be used to describe a number of biological and chemical systems. In general, the equation $\ddot{y}=0$ arises from the variation of the Lagrangian $L=f(y')=\frac{1}{2}y'^2\equiv T$, where T is the kinetic energy of the body (assumed of mass unity). We observe that, for special correlations between parameters and for suitable coordinates changes, the same equation of motion is obtained. Thus, a dynamical system may have more than one Lagrangian function, in agreement with the outcomes of Musielak et al. (2020a) and Musielak et al. (2020b). It is notable that Eq. (19) may be reduced after the same change of coordinate to an equation of the form $y''+ay+by^3=0,(a,b)\in\mathrm{R}$ that describes quartic anharmonic oscillators with quartic potential used in various fields of theoretical physics ranging from atomic to plasma physics and quantum mechanics (Enjieu Kadji et al., 2008; Mickens, 1995; Porwal & Vyas, 2008; Turbiner & Shruyak, 2023). Its solution may be obtained using the improved formulations of the Lindstedt–Poincaré perturbation method (Srivastava & Vishwamittar, 2019).

2.3. The case of the standard Lagrangian $L_3 = e^{\beta x} e^{\varepsilon L}$

For this type of NSL, the associated Euler–Lagrange equation is given by:

$$\beta + \varepsilon \frac{\partial L}{\partial x} - \varepsilon \left(\beta \dot{x} \frac{\partial L}{\partial \dot{x}} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) + \varepsilon^2 \frac{\partial L}{\partial \dot{x}} \frac{\mathrm{d}L}{\mathrm{d}t} \right) = 0. \tag{24}$$

To illustrate, we let $L = \dot{x} + e^x$ which gives the following equation of motion:

$$\varepsilon^3 \ddot{x} + \varepsilon^3 \dot{x} e^x + \varepsilon \beta \dot{x} - \varepsilon e^x - \beta = 0. \tag{25}$$

The change of variable $z = x - \varepsilon^{-2}$ converts Eq. (25) to:

$$\ddot{z} + \dot{z} e^{z + \varepsilon^{-2}} + \varepsilon^{-2} \beta \dot{z} + \varepsilon^{-3} \beta \left(\varepsilon^{-1} - 1 \right) + \varepsilon^{-2} = 0, \tag{26}$$

which for $\beta(\varepsilon^{-1} - 1) + \varepsilon = 0$ is comparable to the SIR equation obtained in the SIR model (Harko et al., 2014; Yoshida, 2022). The NSL function serves as a vital element in describing biological dynamical systems as well.

3. HIGHER-ORDER STANDARD AND NON-STANDARD LAGRANGIANS

In this section, we will show that modified Helmholtz equation and NOQD may be derived from higher-order standard and NSL. The three previous types of Lagrangians $L_1 = e^{\beta x} L$, $L_2 = e^{\beta x} L^{1+\varepsilon}$ and $L_3 = e^{\beta x} e^{\varepsilon L}$ with higher-order Lagrangians $L(\ddot{x}, \dot{x}, x, t)$ will be discussed accordingly. The following subsections will be analysed:

3.1. The case of the standard Lagrangian $L_1 = e^{\beta x} L$

Defining now the functional:

$$S = \int_a^b e^{\beta x} L(\ddot{x}, \dot{x}, x, t) \mathrm{d}t \equiv \int_a^b e^{\beta x} L \mathrm{d}t, \tag{27}$$

it is easy to prove that the associated Euler–Lagrange equation takes the following form:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) + \beta L + 2\beta \dot{x} \frac{\mathrm{d}}{dift} \left(\frac{\partial L}{\partial \ddot{x}} \right) + \beta \ddot{x} \frac{\partial L}{\partial \ddot{x}} + \beta^2 \dot{x}^2 \frac{\partial L}{\partial \ddot{x}} - \beta \dot{x} \frac{\partial L}{\partial \dot{x}} = 0. \tag{28}$$

To illustrate, we consider the HONSL:

$$L(\ddot{x}, \dot{x}, x, t) = \dot{x} - \ddot{x} - m^2 x. \tag{29}$$

For the given Lagrangian, the corresponding Euler–Lagrange Eq. (28) gives the following nonlinear oscillator with quadratic damping:

$$2\beta \ddot{x} + \beta^2 \dot{x}^2 + \beta m^2 x + m^2 = 0, \tag{30}$$

where the solution is also given in terms of the Jacobi elliptic function cn by:

$$x(t) = -\frac{1}{\beta} + D + E \mathrm{cn}^2(\pm W t + F, L^2), \tag{31}$$

where W is the frequency, L is the modulus of the function and D, E, F are real constants. Therefore, HONSL is able as well to derive harmonic oscillator with quadratic damping. Again, the change of variable $\mathrm{d}y = e^{\int f(t) \mathrm{d}t} \mathrm{d}t$ may be used to eliminate the quadratic term.

3.2. The case of the standard Lagrangian $L_2 = e^{\beta x} L^{1+\varepsilon}$

For the functional:

$$S = \int_a^b e^{\beta x} L^{1+\varepsilon}(\ddot{x}, \dot{x}, x, t)\mathrm{d}t \equiv \int_a^b e^{\beta x} L^{1+\varepsilon}\mathrm{d}t, \tag{32}$$

the following Euler–Lagrange equation holds:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) + 2\beta\dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{x}}\right) + \beta\ddot{x}\frac{\partial L}{\partial \ddot{x}} + \beta^2\dot{x}^2\frac{\partial L}{\partial \ddot{x}} - \beta\dot{x}\frac{\partial L}{\partial \dot{x}} + \frac{\beta}{1+\varepsilon}L$$
$$+\frac{\varepsilon}{L}\left(2\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{x}}\right) + \frac{\mathrm{d}^2 L}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}} + \frac{\varepsilon - 1}{L}\left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)^2\frac{\partial L}{\partial \ddot{x}} + 2\beta\dot{x}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}} - \frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{33}$$

Here:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \dot{x}\frac{\partial L}{\partial x} + \ddot{x}\frac{\partial L}{\partial \dot{x}}, \tag{34}$$

and

$$\frac{\mathrm{d}^2 L}{\mathrm{d}t^2} = \frac{\partial^2 L}{\partial t^2} + \ddot{x}\frac{\partial L}{\partial x} + x\frac{\partial L}{\partial \dot{x}} + \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}\right) + \ddot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right). \tag{35}$$

To illustrate, we choose the time-independent NSL:

$$L = \xi\ddot{x} + m^2 x, \tag{36}$$

where $\xi \in$ R. Inserting into Eq. (33) and using Eqs. (34) and (35) and setting for convenience$\varepsilon = 1$, we obtain the following nonlinear differential equation:

$$2\xi m^2\ddot{x} + \beta\xi^2\ddot{x}^2 + \beta\xi m^2 x\ddot{x} + \beta^2\xi^2\dot{x}^2\ddot{x} + \beta^2\xi m^2 x\dot{x}^2 + 2\beta\xi m^2\dot{x}^2 + m^4 x = 0. \tag{37}$$

This equation belongs to the class of 2^{nd}-order nonlinear Duffing differential equations which, for specific values of the parameter, was proved to hold periodic solutions (Evéquoz & Weth, 2014; Liu, 2003; Loud, 1967; Mandel et al., 2017; G. Wang & Cheng, 2007). If, for instance, $m^2 = 0$, Eq. (37) is reduced to $\ddot{x}(\ddot{x} + \beta\dot{x}^2) = 0$, i.e. $\ddot{x} = 0$ (free particle) or $\ddot{x} + \beta\dot{x}^2 = 0$ which gives $x(t) = \frac{1}{\beta}\log(\beta t + c_1) + c_2$, $c_1, c_2, ...$ are constants of integrations. The change of variable $\mathrm{d}y = e^{\int f(t)\mathrm{d}t}\mathrm{d}t$may also convert $\ddot{x} + \beta\dot{x}^2 = 0$ to $\ddot{y} = 0$. Hence, we may obtain the same equation of motion starting from a different Lagrangian. If, for instance, $\beta\xi << 1$, then Eq. (31) is reduced to $2\xi\ddot{x} + m^2 x = 0$ which describes harmonic oscillators with a quadratic potential. Selecting $L = \xi\ddot{x}$ gives rise to the following dynamical equation:

$$\xi\frac{2+\varepsilon}{1+\varepsilon}\ddot{x} + \beta\xi\dot{x}^2 = 0. \tag{38}$$

The change of variable $\mathrm{d}y = e^{\int f(t)\mathrm{d}t}\mathrm{d}t$ gives $\ddot{y} = 0$.

3.3. The case of the standard Lagrangian $L_3 = e^{\beta x} e^{\varepsilon L}$

For the functional:

$$S = \int_a^b e^{\beta x} e^{\varepsilon L(\ddot{x}, \dot{x}, x, t)}\mathrm{d}t, \tag{39}$$

the following Euler–Lagrange equation holds:

$$\beta+\varepsilon\frac{\partial L}{\partial x}-\varepsilon\left(\beta\dot{x}\frac{\partial L}{\partial\dot{x}}+\varepsilon\frac{dL}{dt}\frac{\partial L}{\partial\dot{x}}+\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)\right)+\varepsilon^2\left(\beta\dot{x}\frac{\partial L}{\partial\ddot{x}}+\varepsilon\frac{dL}{dt}\frac{\partial L}{\partial\ddot{x}}+\frac{d}{dt}\left(\frac{\partial L}{\partial\ddot{x}}\right)\right)\frac{dL}{dt}$$
$$+\varepsilon\left(\beta\ddot{x}\frac{\partial L}{\partial\ddot{x}}+\beta\dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\ddot{x}}\right)+\varepsilon\frac{\mathrm{d}^2L}{\mathrm{d}t^2}\frac{\partial L}{\partial\ddot{x}}+\varepsilon\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\ddot{x}}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial\ddot{x}}\right)\right) \tag{40}$$
$$+\varepsilon\beta\dot{x}\left(\beta\dot{x}\frac{\partial L}{\partial\ddot{x}}+\varepsilon\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial\ddot{x}}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\ddot{x}}\right)\right)=0.$$

By letting $L=\ddot{x}+e^x$, we find:

$$\ddot{x}+(1+\varepsilon e^x)\,\dot{x}^2+\frac{1}{\varepsilon}=0. \tag{41}$$

Performing the change of coordinate $\mathrm{d}y=e^{(x+\varepsilon e^x)}\mathrm{d}x$, Eq. (38) is reduced to:

$$\ddot{x}+(1+\varepsilon e^x)\,\dot{x}^2+\frac{1}{\varepsilon}=0. \tag{42}$$

Letting where $y=e^{\delta z},\delta\in\tilde{}$ reduces Eq. (39) to:

$$\ddot{z}+\dot{z}^2+\frac{1}{\delta\varepsilon}(\ln\varepsilon+\delta z)=0. \tag{43}$$

In particular, for $\varepsilon=1$, we obtain the NOQD equation $\ddot{z}+\dot{z}^2+z=0$. Hence, one may argue that a family of nonlinear oscillators used in several scientific fields of research may also be derived from HONSL. Undoubtedly, one may pick a large number of NSL and HONSL (El-Nabulsi, 2018a, 2018b, 2023; El-Nabulsi & Anukool, 2022, 2023, 2024; El-Nabulsi & Golmankhaneh, 2022), so far the Lagrangians introduced in this study are motivating.

4. DISCUSSION

In the work done by Musielak (2008), equations of motion of the forms $\ddot{x}+f(x)\dot{x}^n+g(x)w(x)=0, n=1,2,3,\ldots$ and $\ddot{x}+g(t)\dot{x}^m+k(t)W(x)=0, m=1,2,3,\ldots$ are obtained from NSL with variable coefficients. For $m=n=1$, damped Duffing oscillator, damped Duffing–van der Pol system, the Lienard-type equation, and Lienard and other types of oscillators are obtained. For $m=n=2$, a number of dynamical equations with nonlinear damping are also obtained accordingly (Cariñena et al., 2004; Hayashi, 1985; Jiménez et al., 2005; Matzner & Shepley, 1991; Stuckens & Kobe, 1986; Trueba et al., 2000). In the present work, a comparable set of equations of the Duffing–van der Pol type equation, including the 2^{nd}–order nonlinear Duffing differential equations, has been obtained by using a simpler and more direct approach without recourse to the Lie symmetry methods (Hydon, 2000) or to the transformation technique introduced in Udwadia and Cho (2014). Although our NSL approach employed herein is simple and trouble-free, the solutions of higher-order Duffing differential equations under more general conditions need much more work. Nevertheless, the idea of obtaining 2^{nd}-order nonlinear differential equations from 2^{nd}-order NSL is a motivating idea. Notably, equations of type (21) belong to the class of Lienard–type equation where their generic form is given by $\ddot{x}+f(x)\dot{x}+G(x)=0$. For the case of Eq. (21), the elimination of both the pure quadratic term and the hybrid term by adjusting the coefficients in the Lagrangian gives rise to an equation of type $\ddot{x}+f(x)\dot{x}+G(x)=0$. The Lienard–type equation is motivating since it is used for the description of different phenomena in physics, biology, mechanics, and engineering (Cariñena & Guha, 2019; Guckenheimer & Holmes, 1983; Kudryashov & Sinelshchikov, 2017; Zaitsev & Polyanin, 2002). An analogy between a system with a quadratic in velocity damping $\ddot{x}+f(x)\dot{x}^2+g(x)=0$ and a position-dependent mass system has also been obtained. The methodology of this paper is rather new and can be used to derive a family of highly nonlinear higher-order differential equations characterized by quadratic nonlinearities. It is therefore motivating to extend our approach to study the oscillatory behavior of higher-order differential equations subject to specific initial conditions (Benaicha & Haddouchi, 2016; El-Nabulsi et al., 2020; Ma & Chen, 2011; Moaaz et al., 2020; Q. Wang et al., 2013). The differential equations $\ddot{x} + f(x)\dot{x}^n + g(x)w(x) = 0, n = 1, 2, 3, ...$ and $\ddot{x} + g(t)\dot{x}^m + k(t)W(x) = 0, m = 1, 2, 3, ...$ exemplify a class of nonlinear dynamical systems that arise in various areas of physics and applied mathematics, including nonlinear oscillations, dissipative systems, and certain models in field theory. Its significance lies in the interplay between nonlinearity and damping-like effects introduced through terms such as $\dot{x}^2$, which make the system behavior richer and more complex than classical linear oscillators. Nonstandard Lagrangians play a crucial role in deriving such equations, as they allow the formulation of dynamical systems whose equations of motion do not follow from conventional kinetic-minus-potential energy forms. By extending the Lagrangian framework, one can systematically construct equations incorporating velocity-dependent forces or other nontraditional interactions, providing a variational foundation for studying these nonlinear phenomena and revealing hidden symmetries or conserved quantities that might otherwise remain obscure. Nonstandard Lagrangians are particularly relevant in this context because they extend the classical variational approach, allowing the systematic derivation of equations of motion that include nontraditional interactions, velocity-dependent forces, or nonconservative terms. By employing nonstandard Lagrangians, one can identify conserved quantities, hidden symmetries, and integrability conditions, offering a deeper theoretical understanding of complex dynamical systems and opening pathways for both analytical and numerical investigations in advanced physics and engineering applications.

5. CONCLUSIONS

To conclude, motivated by some recent advances in geometrical mechanics theories, we extended the Lagrangian formalism by introducing three types of non-standard Lagrangians. We have proved that the modified Helmholtz differential equation with linear damping, pure quadratic damping and hybrid damping may be obtained from these special types of Lagrangians. An analogy with Liénard type equation and position-dependent mass systems was given. Besides, it was observed that for some particular cases, a simple change of coordinates may convert the nonlinear oscillators to differential type-equations used in modeling SIR biological systems. We have also proved that the modified Helmholtz differential equation may also be obtained from higher-order derivatives of standard and non-standard Lagrangians. We hope that the present approach will be used to derive other equations of motion subject to nonlinearities that could have several applications in sciences and engineering subject to chaos and fluctuating forces.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CREDIT AUTHOR STATEMENT

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REFERENCES

Aldaya, V., & Azcarraga, J. A. d. (1992). Higher-order Hamiltonian formalism in field theory. *Journal of Physics A: Mathematical and General*, *13*(8), 2545–2551. https://doi.org/10.1088/0305-4470/13/8/004

Andrianov, A. A., Cannada, F., & Loffe, M. V. (1996). Contraction of higher order derivative SUSY algebras in quantum mechanics with large energy gap. *Modern Physics Letters A*, *11*(17), 1417–1428. https://doi.org/10.1142/s0217732396001429

Arnold, V. I. (1978). *Mathematical methods of classical mechanics*. Springer New York. https://doi.org/10.1007/978-1-4757-1693-1

Bateman, H. (1931). On dissipative systems and related variational principles. *Physical Review*, *38*(4), 815–819. https://doi.org/10.1103/physrev.38.815

Benaicha, S., & Haddouchi, F. (2016). Positive solutions of a nonlinear fourth-order integral boundary value problem. *Annals of West University of Timisoara - Mathematics and Computer Science*, *54*(1), 73–86. https://doi.org/10.1515/awutm-2016-0005

Bender, C. M., & Mannheim, P. D. (2008). No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model. *Physical Review Letters*, *100*(11), 110402. https://doi.org/10.1103/physrevlett.100.110402

Bender, C. M., Holm, D. D., & Hook, D. W. (2007). Complex trajectories of a simple pendulum. *Journal of Physics A: Mathematical and Theoretical*, *40*(3), F81–F89. https://doi.org/10.1088/1751-8113/40/3/f01

Benenti, S. (2005). The Lagrangian and Hamiltonian formulations for a special class of non-conservative systems. In G. Gaeta (Ed.), *Symmetry and perturbation theory* (pp. 43–50). World Scientific. https://doi.org/10.1142/9789812702142_0006

Berberan-Santos, M. N., Bodunov, E. N., & Pogliani, L. (2005). Classical and quantum study of the motion of a particle in a gravitational field. *Journal of Mathematical Chemistry*, *37*(2), 101–115. https://doi.org/10.1007/s10910-004-1443-y

Borges, L. H. C., Barone, F. A., de Melo, C. A. M., & Barone, F. E. (2019). Higher order derivative operators as quantum corrections. *Nuclear Physics B*, *944*, 114634. https://doi.org/10.1016/j.nuclphysb.2019.114634

Byrd, P. F., & Friedman, M. D. (1954). *Handbook of elliptic integrals for engineers and physicists*. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-52803-3

Caldirola, P. (1941). Forze non conservative nella meccanica quantistica. *Il Nuovo Cimento*, *18*(9), 393–400. https://doi.org/10.1007/bf02960144

Cariñena, J. F., & Núñez, J. F. (2016a). Geometric approach to dynamics obtained by deformation of Lagrangians. *Nonlinear Dynamics*, *83*(1–2), 457–461. https://doi.org/10.1007/s11071-015-2340-6

Cariñena, J. F., & Núñez, J. F. (2016b). Geometric approach to dynamics obtained by deformation of time-dependent Lagrangians. *Nonlinear Dynamics*, *86*(2), 1285–1291. https://doi.org/10.1007/s11071-016-2964-1

Cariñena, J. F., Gheorghiu, I., Martínez, E., & Santos, P. (2014). Conformal Killing vector fields and a virial theorem. *Journal of Physics A: Mathematical and Theoretical*, *47*(46), 465206. https://doi.org/10.1088/1751-8113/47/46/465206

Cariñena, J. F., & Guha, P. (2019). Nonstandard Hamiltonian structures of the Liénard equation and contact geometry. *International Journal of Geometric Methods in Modern Physics*, *16*(supp01), 1940001. https://doi.org/10.1142/s0219887819400012

Cariñena, J. F., Rañada, M. F., & Santander, M. (2005). Lagrangian formalism for nonlinear second-order Riccati systems: One-dimensional integrability and two-dimensional superintegrability. *Journal of Mathematical Physics*, *46*(6), 062703. https://doi.org/10.1063/1.1920287

Cariñena, J. F., Rañada, M. F., Santander, M., & Senthilvelan, M. (2004). A non-linear oscillator with quasi-harmonic behaviour: Two- and n-dimensional oscillators. *Nonlinearity*, *17*(5), 1941–1963. https://doi.org/10.1088/0951-7715/17/5/019

Cieśliński, J. L., & Nikiciuk, T. (2010). A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients. *Journal of Physics A: Mathematical and Theoretical*, *43*(17), 175205. https://doi.org/10.1088/1751-8113/43/17/175205

Cline, D. (2018). *Variational principles in classical mechanics* (2nd ed.). University of Rochester River Campus Libraries University of Rochester Press.

Crampin, M., Mestdag, T., & Sarlet, W. (2010). On the generalized Helmholtz conditions for Lagrangian systems with dissipative forces. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, *90*(6), 502–508. https://doi.org/10.1002/zamm.200900327

Crampin, M., & Sarlet, W. (2001). A class of nonconservative Lagrangian systems on Riemannian manifolds. *Journal of Mathematical Physics*, *42*(9), 4313–4326. https://doi.org/10.1063/1.1388030

Cruz y Cruz, S., Negro, J., & Nieto, L. M. (2008). On position-dependent mass harmonic oscillators. *Journal of Physics: Conference Series*, *128*, 012053. https://doi.org/10.1088/1742-6596/128/1/012053

Cruz y Cruz, S., & Rosas-Ortiz, O. (2009). Position-dependent mass oscillators and coherent states. *Journal of Physics A: Mathematical and Theoretical*, *42*(18), 185205. https://doi.org/10.1088/1751-8113/42/18/185205

Cveticanin, L. (2009). Oscillator with strong quadratic damping force. *Publications de l'Institut Math?matique (Belgrade)*, *85*(99), 119–130. https://doi.org/10.2298/pim0999119c

Davachi, N., & Musielak, Z. E. (2019). Generalized non-standard lagrangians. *Journal of Undergraduate Reports in Physics*, *29*(1), 100004. https://doi.org/10.1063/1.5129244

Dong, S. H., Pena, J. J., Pacheco-García, C., & Garcia-Ravelo, J. (2007). Algebraic approach to the position-dependent mass Schrödinger for a singular oscillator. *Modern Physics Letters A*, *22*(14), 1039–1045. https://doi.org/10.1142/s0217732307021470

El-Nabulsi, R. A. (2009). Complexified quantum field theory and "mass without mass" from multidimensional fractional actionlike variational approach with dynamical fractional exponents. *Chaos, Solitons & Fractals*, *42*(4), 2384–2398. https://doi.org/10.1016/j.chaos.2009.03.115

El-Nabulsi, R. A. (2012). Nonlinear dynamics with nonstandard Lagrangians. *Qualitative Theory of Dynamical Systems*, *12*(2), 273–291. https://doi.org/10.1007/s12346-012-0074-0

El-Nabulsi, R. A. (2013a). Modified Proca equation and modified dispersion relation from a power-law Lagrangian functional. *Indian Journal of Physics*, *87*(5), 465–470. https://doi.org/10.1007/s12648-012-0237-5

El-Nabulsi, R. A. (2013b). Non-standard fractional Lagrangians. *Nonlinear Dynamics*, *74*(1–2), 381–394. https://doi.org/10.1007/s11071-013-0977-6

El-Nabulsi, R. A. (2013c). Quantum field theory from an exponential action functional. *Indian Journal of Physics*, *87*(4), 379–383. https://doi.org/10.1007/s12648-012-0187-y

El-Nabulsi, R. A. (2014). Electrodynamics of relativistic particles through non-standard Lagrangians. *Journal of Atomic and Molecular Sciences*, *5*(3), 268–278. https://doi.org/10.4208/jams.040214.061414a

El-Nabulsi, R. A. (2015a). Non-standard power-law Lagrangians in classical and quantum dynamics. *Applied Mathematics Letters*, *43*, 120–127. https://doi.org/10.1016/j.aml.2014.12.002

El-Nabulsi, R. A. (2017). Complex backward-forward derivative operator in non-local-in-time Lagrangian mechanics. *Qualitative Theory of Dynamical Systems*, *16*(2), 223–234. https://doi.org/10.1007/s12346-016-0187-y

El-Nabulsi, R. A. (2018a). Gravitational field as a pressure force from logarithmic Lagrangians and non-standard Hamiltonians: The case of the stellar halo of the Milky Way. *Communications in Theoretical Physics*, *69*(3), 233–240. https://doi.org/10.1088/0253-6102/69/3/233
El-Nabulsi, R. A. (2018b). The Hamilton-Jacobi analysis of powers of singular Lagrangians: A connection between the modified Schrödinger and the Navier-Stokes equations. *Qualitative Theory of Dynamical Systems*, *17*(3), 583–608. https://doi.org/10.1007/s12346-017-0257-9
El-Nabulsi, R. A. (2020a). Inverse-power potentials with positive-bound energy spectrum from fractal, extended uncertainty principle and position-dependent mass arguments. *The European Physical Journal Plus*, *135*(9), 683. https://doi.org/10.1140/epjp/s13360-020-00717-w
El-Nabulsi, R. A. (2020b). Non-standard magnetohydrodynamics equations and their implications in sunspots. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *476*(2241), 20200190. https://doi.org/10.1098/rspa.2020.0190
El-Nabulsi, R. A. (2020c). Nonlocal thermodynamics properties of position-dependent mass particle in magnetic and Aharonov-Bohm flux fields. *Few-Body Systems*, *61*(4), 37. https://doi.org/10.1007/s00601-020-01569-x
El-Nabulsi, R. A. (2023). Damped quantum wave equation from non-standard Lagrangians and damping terms removal. *Waves in Random and Complex Media*, *33*(4), 1027–1044. https://doi.org/10.1080/17455030.2021.1896052
El-Nabulsi, R. A., & Anukool, W. (2022). A new approach to nonlinear quartic oscillators. *Archive of Applied Mechanics*, *92*(1), 351–362. https://doi.org/10.1007/s00419-021-02062-5
El-Nabulsi, R. A., & Anukool, W. (2023). Orbital dynamics, chaotic orbits and Jacobi elliptic functions. *The Journal of the Astronautical Sciences*, *70*(1), 1. https://doi.org/10.1007/s40295-023-00367-x
El-Nabulsi, R. A., & Anukool, W. (2024). Higher-order nonlocal approach to classical and quantum mechanics through non-standard Lagrangians: The case of quantum wells and the quantum states of neutron in earth's gravitational field. *Modern Physics Letters A*, *39*(13), 2450054. https://doi.org/10.1142/s0217732324500548
El-Nabulsi, R. A., Soulati, T. A., & Rezazadeh, H. (2013). Non-standard complex Lagrangian dynamics. *Journal of Advanced Research in Dynamical and Control Systems*, *5*(1), 50–62.
El-Nabulsi, R. A., & Wu, G.-c. (2012). Fractional complexified field theory from Saxena-Kumbhat fractional integral, fractional derivative of order (α,β) and dynamical fractional integral exponent. *African Diaspora Journal of Mathematics*, *13*, 45–61.
El-Nabulsi, R. A. (2013d). Nonstandard Lagrangian cosmology. *Journal of Theoretical and Applied Physics*, *7*(1), 58. https://doi.org/10.1186/2251-7235-7-58
El-Nabulsi, R. A. (2015b). Non-standard Lagrangians with higher-order derivatives and the Hamiltonian formalism. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, *85*(2), 247–252. https://doi.org/10.1007/s40010-014-0192-y
El-Nabulsi, R. A., & Golmankhaneh, A. K. (2022). Non-standard and fractal electrodynamics in Finsler-Randers space. *International Journal of Geometric Methods in Modern Physics*, *19*(06), 2250080. https://doi.org/10.1142/s0219887822500803
El-Nabulsi, R. A., Moaaz, O., & Bazighifan, O. (2020). New results for oscillatory behavior for fourth-order differential equations. *Symmetry*, *12*(1), 136. https://doi.org/10.3390/sym12010136
Enjieu Kadji, H. G., Nana Nbendjo, B. R., Chabi Orou, J. B., & Talla, P. K. (2008). Nonlinear dynamics of plasma oscillations modeled by an anharmonic oscillator. *Physics of Plasmas*, *15*(3), 032308. https://doi.org/10.1063/1.2841032
Epstein, I. R., & Pojman, J. A. (1998, November). *An introduction to nonlinear chemical dynamics: Oscillations, waves, patterns and complexity and chaos*. Oxford University Press. https://doi.org/10.1093/oso/9780195096705.001.0001

Eshghi, M., Sever, R., & Ikhdair, S. M. (2018). Energy states of the Hulthén plus Coulomb-like potential with position-dependent mass function in external magnetic fields. *Chinese Physics B*, *27*(2), 020301. https://doi.org/10.1088/1674-1056/27/2/020301

Evéquoz, G., & Weth, T. (2014). Real solutions to the nonlinear Helmholtz equation with local nonlinearity. *Archive for Rational Mechanics and Analysis*, *211*(2), 359–388. https://doi.org/10.1007/s00205-013-0664-2

Fangnon, R., Ainamon, C., Monwanou, A. V., Miwadinou, C. H., & Chabi Orou, J. B. (2020). Nonlinear dynamics of the quadratic-damping Helmholtz oscillator (K. Rajagopal, Ed.). *Complexity*, *2020*, 1–17. https://doi.org/10.1155/2020/8822534

Fay, T. H. (2012). Quadratic damping. *International Journal of Mathematical Education in Science and Technology*, *43*(6), 789–803. https://doi.org/10.1080/0020739x.2011.622806

Gràcia, X., Pons, J. M., & Román-Roy, N. (1991). Higher-order Lagrangian systems: Geometric structures, dynamics, and constraints. *Journal of Mathematical Physics*, *32*(10), 2744–2763. https://doi.org/10.1063/1.529066

Gubbiotti, G., & Nucci, M. C. (2015). Quantization of quadratic Liénard-type equations by preserving Noether symmetries. *Journal of Mathematical Analysis and Applications*, *422*(2), 1235–1246. https://doi.org/10.1016/j.jmaa.2014.09.045

Guckenheimer, J., & Holmes, P. (1983). *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer New York. https://doi.org/10.1007/978-1-4612-1140-2

Halder, A. K., Paliathanasis, A., & Leach, P. G. L. (2019). Singularity analysis of a variant of the Painlevé-Ince equation. *Applied Mathematics Letters*, *98*, 70–73. https://doi.org/10.1016/j.aml.2019.05.042

Harko, T., Lobo, F. S. N., & Mak, M. K. (2014). Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates. *Applied Mathematics and Computation*, *236*, 184–194. https://doi.org/10.1016/j.amc.2014.03.030

Hayashi, C. (1985). *Nonlinear oscillations in physical systems*. Princeton University Press.

Hong-Xia, Z., Shan-Jun, M., & Yong, S. (2008). Higher-order Lagrangian equations of higher-order motive mechanical system. *Communications in Theoretical Physics*, *49*(2), 479–481. https://doi.org/10.1088/0253-6102/49/2/47

Hu, H. (2006). Solution of a quadratic nonlinear oscillator by the method of harmonic balance. *Journal of Sound and Vibration*, *293*(1–2), 462–468. https://doi.org/10.1016/j.jsv.2005.10.002

Hu, H. (2007). Solution of a mixed-parity nonlinear oscillator: Harmonic balance. *Journal of Sound and Vibration*, *299*(1–2), 331–338. https://doi.org/10.1016/j.jsv.2006.06.046

Hydon, P. E. (2000, January). *Symmetry methods for differential equations: A beginner's guide*. Cambridge University Press. https://doi.org/10.1017/cbo9780511623967

Jiang, J., Feng, Y., & Xu, S. (2019). Noether's symmetries and its inverse for fractional logarithmic Lagrangian systems. *Journal of Systems Science and Information*, *7*(1), 90–98. https://doi.org/10.21078/jssi-2019-090-09

Jiménez, J. L., del Valle, G., & Campos, I. (2005). A canonical treatment of some systems with friction. *European Journal of Physics*, *26*(5), 711–725. https://doi.org/10.1088/0143-0807/26/5/004

Kamalov, T. (2013). Classical and quantum-mechanical axioms with the higher time derivative formalism. *Journal of Physics: Conference Series*, *442*, 012051. https://doi.org/10.1088/1742-6596/442/1/012051

Kamalov, T. F. (2009). A model of extended mechanics and nonlocal hidden variables for quantum theory. *Journal of Russian Laser Research*, *30*(5), 466–471. https://doi.org/10.1007/s10946-009-9098-6

Kamalov, T. F. (2020). Quantum corrections of Newton's law of motion. *Symmetry*, *12*(1), 63. https://doi.org/10.3390/sym12010063

Kanai, E. (1948). On the quantization of the dissipative systems. *Progress of Theoretical Physics*, *3*(4), 440–442. https://doi.org/10.1143/ptp/3.4.440

Kudryashov, N. A., & Sinelshchikov, D. I. (2017). New non-standard Lagrangians for the Liénard-type equations. *Applied Mathematics Letters*, *63*, 124–129. https://doi.org/10.1016/j.aml.2016.07.028

Leon, D., & Rodrigues, M. P. R. (1985). *Generalized classical mechanics and field theory*. North-Holland.

Liu, W. B. (2003). The existence of periodic solutions for high order Duffing equations. *Acta Mathematica Sinica*, *46*, 49–56.

Llibre, J. (2023). Dynamics of the painlevé-ince equation. *Results in Mathematics*, *78*(1), 11. https://doi.org/10.1007/s00025-022-01783-5

Loud, W. S. (1967). Periodic solutions of nonlinear differential equation of Duffing types. In *Differential and functional equations* (pp. 199–224). Benjamin.

Ma, R., & Chen, T. (2011). Existence of positive solutions of fourth-order problems with integral boundary conditions. *Boundary Value Problems*, *2011*, 1–17. https://doi.org/10.1155/2011/297578

Madison, J. V. (2012). Isenthalpic oscillations with quadratic damping in saturated two-phase fluids. *WIT Transactions on Engineering Sciences*, *74*, 393–401. https://doi.org/10.2495/afm120351

Magal, P., & Ruan, S. (2014). Susceptible-infectious-recovered models revisited: From the individual level to the population level. *Mathematical Biosciences*, *250*, 26–40. https://doi.org/10.1016/j.mbs.2014.02.001

Mahomed, F. M., & Leach, P. G. L. (1985). The linear symmetries of a nonlinear differential equation. *Quaestiones Mathematicae*, *8*(3), 241–274. https://doi.org/10.1080/16073606.1985.9631915

Mandal, B. P., & Mahajan, S. S. (2015). Complex classical mechanics of a QES potential. *Communications in Theoretical Physics*, *64*(4), 425–432. https://doi.org/10.1088/0253-6102/64/4/425

Mandel, R., Montefusco, E., & Pellacci, B. (2017). Oscillating solutions for nonlinear Helmholtz equations. *Zeitschrift für angewandte Mathematik und Physik*, *68*(6), 121. https://doi.org/10.1007/s00033-017-0859-8

Matzner, R. A., & Shepley, L. C. (1991). *Classical mechanics*. Prentice-Hall.

Mei, F. X., Liu, D., & Luo, Y. (1991). *Advanced analytical mechanics*. Beijing Institute of Technology Press.

Mickens, R. E. (1995). *Oscillations in planar dynamic systems*. World Scientific. https://doi.org/10.1142/2778

Milne-Thomson, L. M. (1950). *Jacobian elliptic function tables*. Dover Publications Inc.

Miwadinou, C. H., Monwanou, A. V., Hinvi, L. A., & Orou, J. B. C. (2018). Effect of amplitude modulated signal on chaotic motions in a mixed Rayleigh-Lienard oscillator. *Chaos, Solitons & Fractals*, *113*, 89–101. https://doi.org/10.1016/j.chaos.2018.05.021

Miwadinou, C. H., Monwanou, A. V., Koukpemedji, A. A., Kpomahou, Y. J. F., & Chabi Orou, J. B. (2018). Chaotic motions in forced mixed Rayleigh-Lienard oscillator with external and parametric periodic-excitations. *International Journal of Bifurcation and Chaos*, *28*(03), 1830005. https://doi.org/10.1142/s0218127418300057

Moaaz, O., El-Nabulsi, R. A., & Bazighifan, O. (2020). Oscillatory behavior of fourth-order differential equations with neutral delay. *Symmetry*, *12*(3), 371. https://doi.org/10.3390/sym12030371

Musielak, Z. E. (2008). Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients. *Journal of Physics A: Mathematical and Theoretical*, *41*(5), 055205. https://doi.org/10.1088/1751-8113/41/5/055205

Musielak, Z. E. (2009). General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems. *Chaos Solitons and Fractals*, *42*(5), 2645–2652. https://doi.org/10.1016/j.chaos.2009.03.171

Musielak, Z. E., Davachi, N., & Rosario-Franco, M. (2020a). Lagrangians, Gauge transformations and Lie groups for commutative semigroup of differential equations. *Journal of Applied Mathematics*, *2020*, 1–11. https://doi.org/10.1155/2020/3170130

Musielak, Z. E., Roy, D., & Swift, L. D. (2008). Method to derive Lagrangian and Hamiltonian for a nonlinear dynamical system with variable coefficients. *Chaos, Solitons and Fractals*, *38*(3), 894–902. https://doi.org/10.1016/j.chaos.2007.06.076

Musielak, Z. E., Davachi, N., & Rosario-Franco, M. (2020b). Special functions of mathematical physics: A unified Lagrangian formalism. *Mathematics*, *8*(3), 379. https://doi.org/10.3390/math8030379

Mustafa, O. (2015). Position-dependent mass Lagrangians: Nonlocal transformations, Euler-Lagrange invariance and exact solvability. *Journal of Physics A: Mathematical and Theoretical*, *48*(22), 225206. https://doi.org/10.1088/1751-8113/48/22/225206

Nayfey, A. H., & Mook, D. T. (1979). *Nonlinear oscillations*. Wiley.

Olabodé, D. L., Miwadinou, C. H., Monwanou, V. A., & Chabi Orou, J. B. (2019). Effects of passive hydrodynamics force on harmonic and chaotic oscillations in nonlinear chemical dynamics. *Physica D: Nonlinear Phenomena*, *386–387*, 49–59. https://doi.org/10.1016/j.physd.2018.09.001

Paliathanasis, A., & Duffy, J. (2025). Lagrangians and Newtonian analogs for biological systems. *The European Physical Journal Plus*, *140*(3), 191. https://doi.org/10.1140/epjp/s13360-025-06078-6

Pandey, A., Choudhury, A. G., & Guha, P. (2016). Quadratically damped oscillators with non-linear restoring force. *arXiv:1610.07821*.

Porwal, R., & Vyas, N. S. (2008). Damped quadratic and mixed-parity oscillator response using Krylov-Bogoliubov method and energy balance. *Journal of Sound and Vibration*, *309*(3–5), 877–886. https://doi.org/10.1016/j.jsv.2007.07.028

Poston, T., & Stewart, I. N. (1978). *Catastrophe theory and its applications*. Pitman.

Román-Roy, N. (2009). Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories. *Symmetry, Integrability and Geometry: Methods and Applications*, *5*, 100–124. https://doi.org/10.3842/sigma.2009.100

Saha, A., & Talukdar, B. (2014). Inverse variational problem for non-standard Lagrangians. *Reports on Mathematical Physics*, *73*(3), 299–309. https://doi.org/10.1016/s0034-4877(14)60046-x

Sanjuán, M. A. F. (1999). Effect of nonlinear damping on the universal escape oscillator. *International Journal of Bifurcation and Chaos*, *9*(04), 735–744. https://doi.org/10.1142/s0218127499000523

Sbitnev, V. E. (2009). Bohmian trajectories and the path integral paradigm: Complexified Lagrangian mechanics. *Int. J. Bifurcation & Chaos*, *19*(07), 2335–2346. https://doi.org/10.1142/s0218127409024104

Simon, J. Z. (1990a). Higher-derivative Lagrangians, nonlocality, problems, and solutions. *Physical Review D*, *41*(12), 3720–3733. https://doi.org/10.1103/physrevd.41.3720

Simon, J. Z. (1990b, August). *Higher derivative expansions and non-locality* [phd]. University of California, Santa Barbara.

Skinner, R. (1983). First-order equations of motion for classical mechanics. *Journal of Mathematical Physics*, *24*(11), 2581–2588. https://doi.org/10.1063/1.525653

Skinner, R., & Rusk, R. (1983a). Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus T Q$. *Journal of Mathematical Physics*, *24*(11), 2589–2594. https://doi.org/10.1063/1.525654

Skinner, R., & Rusk, R. (1983b). Generalized Hamiltonian dynamics. II. Gauge transformations. *Journal of Mathematical Physics*, *24*(11), 2595–2601. https://doi.org/10.1063/1.525655

Smith, B. R. (2012). The Quadratically damped oscillator: A case study of a non-linear equation of motion. *American Journal of Physics*, *80*(9), 816–824. https://doi.org/10.1119/1.4729440

Soliman, M., & Thompson, J. M. T. (1992). Effect of damping on the steady state and basin bifurcation patterns of a nonlinear mechanical oscillator. *International Journal of Bifurcation and Chaos*, *2*(01), 81–91. https://doi.org/10.1142/s0218127492000082

Song, J., & Zhang, Y. (2018). Noether's theorems for dynamical systems of two kinds of non-standard Hamiltonians. *Acta Mechanica*, *229*(1), 285–297. https://doi.org/10.1007/s00707-017-1967-4

Spyrou, K. J., Cotton, B., & Gurd, B. (2002). Analytical expressions of capsize boundary of a ship with roll bias in beam waves. *Journal of Ship Research*, *46*(03), 167–174. https://doi.org/10.5957/jsr.2002.46.3.167

Srivastava, S. S., & Vishwamittar. (2019). A comparative study of some analytic approximate methods for the anharmonic oscillator described by cubic quartic potential. *arXiv:1911.08283*. https://doi.org/10.48550/arxiv.1911.08283

Stuckens, C., & Kobe, D. H. (1986). Quantization of a particle with a force quadratic in the velocity. *Physical Review A*, *34*(5), 3565–3567. https://doi.org/10.1103/physreva.34.3565

Suykens, J. A. K. (2009). Extending Newton's law from nonlocal-in-time kinetic energy. *Physics Letters A*, *373*(14), 1201–1211. https://doi.org/10.1016/j.physleta.2009.01.065

Thompson, J. M. T. (1997). Designing against capsize in beam seas: Recent advances and new insights. *Applied Mechanics Reviews*, *50*(5), 307–325. https://doi.org/10.1115/1.3101710

Tiwari, A. K., Pandey, S. N., Senthilvelan, M., & Lakshmanan, M. (2013). Classification of Lie point symmetries for quadratic Liénard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$. *Journal of Mathematical Physics*, *54*(5), 053506. https://doi.org/10.1063/1.4803455

Trueba, J. L., Rams, J., & Sanjuan, M. A. (2000). Analytical estimates of the effect of nonlinear damping in some nonlinear oscillators. *International Journal of Bifurcation and Chaos*, *10*(09), 2257–2267. https://doi.org/10.1142/S0218127400001419

Tshibase, S., & Govinder, K. S. (2025). λ-Symmetries of the Painlevé–Ince equation. *Afrika Matematika*, *36*(4), 154. https://doi.org/10.1007/s13370-025-01361-2

Turbiner, A. V., & Shruyak, E. (2023). On the connection between perturbation theory and new semiclassical expansion in quantum mechanics. *Nuclear Physics B*, *989*, 116117. https://doi.org/10.1016/j.nuclphysb.2023.116117

Udwadia, F. E., & Cho, H. (2014). First integrals and solutions of Duffing-van der Pol type equations. *Journal of Applied Mechanics*, *81*(3), 034501–034504. https://doi.org/10.1115/1.4024673

Vestal, L. C., & Musielak, Z. E. (2021). Bateman oscillators: Caldirola-Kanai oscillators and null Lagrangians and Gauge functions. *Phys.*, *3*(2), 449–458. https://doi.org/10.3390/physics3020030

Vitagliano, L. (2011). On higher derivatives as constraints in field theory: A geometric perspective. *International Journal of Geometric Methods in Modern Physics*, *08*(08), 1687–1693. https://doi.org/10.1142/s0219887811005968

Vitagliano, L. (2010). The Lagrangian-Hamiltonian formalism for higher order field theories. *Journal of Geometry and Physics*, *60*(6–8), 857–873. https://doi.org/10.1016/j.geomphys.2010.02.003

Wang, G., & Cheng, S. S. (2007). Even periodic solutions of higher order Duffing differential equations. *Czechoslovak Mathematical Journal*, *57*(1), 331–343. https://doi.org/10.1007/s10587-007-0063-7

Wang, Q., Guo, Y., & Ji, Y. (2013). Positive solutions for fourth-order nonlinear differential equation with integral boundary conditions. *Discrete Dynamics in Nature and Society*, *2013*, 1–10. https://doi.org/10.1155/2013/684962

Yamamoto, T., & Nath, J. H. (1976). High Reynolds number oscillating flow by cylinders. *Coastal Engineering 1976*, 2321–2340. https://doi.org/10.1061/9780872620834.136

Yong-Fen, Q., Yao-Liang, Z., & Guang-Cai, H. (2002). Integrating factors and conservation theorem for holonomic nonconservative dynamical systems in generalized classical mechanics. *Chinese Physics*, *11*(10), 988–992. https://doi.org/10.1088/1009-1963/11/10/302

Yoshida, N. (2022). Exact solution of the Susceptible-Infectious-Recovered (SIR) epidemic model. *arXiv:2210.00444*. https://doi.org/10.48550/arXiv.2210.00444

Yu, J., Dong, S.-H., & Sun, G.-H. (2004). Series solutions of the Schrödinger equation with position-dependent mass for the Morse potential. *Physics Letters A*, *322*(5–6), 290–297. https://doi.org/10.1016/j.physleta.2004.01.039

Zaitsev, V. F., & Polyanin, A. D. (2002, October). *Handbook of exact solutions for ordinary differential equations*. Chapman; Hall/CRC. https://doi.org/10.1201/9781420035339

Zhang, Y., & Wang, X.-P. (2019). Mei symmetry and invariants of quasi-fractional dynamical systems with non-standard Lagrangians. *Symmetry*, *11*(8), 1061. https://doi.org/10.3390/sym11081061

Zhang, Y., & Zhou, X.-S. (2016). Noether theorem and its inverse for nonlinear dynamical systems with nonstandard Lagrangians. *Nonlinear Dynamics*, *84*(4), 1867–1876. https://doi.org/10.1007/s11071-016-2611-x

Zhou, X. S., & Zhang, Y. (2016). Routh method of reduction for dynamical systems with nonstandard Lagrangians. *Chinese Quarterly of Mechanics*, *37*, 15–21.