

BENDING EVALUATION OF QUINTUPLE-LAYER FG BEAMS WITH SLIDING INTERACTIONS

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Abstract. This paper introduces a model of a quintuple-layer composite beam made of functionally graded materials (FGMs) and shear connectors. A finite element model is developed using the first-order shear deformation theory (FSDT) of Timoshenko, employing a two-node beam element with eleven degrees of freedom per node. The formulation accounts for interlayer sliding interaction. Based on this theory, a computational program is implemented in MATLAB and validated with various benchmark models. The results provide a basis for further work, such as free vibration analysis and dynamic response studies of complex beam structures.

Keywords: bending, quintuple-layer FG beams, sliding interactions, FEM, FSDT.

1. INTRODUCTION

Multi-layered composite beams, particularly those using shear connectors, are becoming increasingly significant as they combine the advantages of layered construction with the benefits of regulated interfacial behavior. The use of shear connectors helps mitigate interlayer slippage and enhances load transmission, which are critical factors for precisely simulating real-world composite structures (Chybiński & Polus, 2023; Paknahad et al., 2018). Modeling the mechanical behavior of these systems is challenging due to the complexity of interlayer interactions, variability in material characteristics, and the need for robust numerical methods.

Numerous studies have investigated the mechanical behavior of functionally graded material (FGM) beams through various theoretical and numerical methodologies, including classical beam theory, higher-order shear deformation theories, and both mesh-based and mesh-free computational techniques. However, the majority of current studies simplify interfacial behavior by supposing perfect bonding or neglecting slip between layers, thereby resulting in inaccuracies in structural analysis and design. For example, Ataei et al. (2019) conducted an experimental investigation into the cyclic behavior of bolted shear connectors, revealing that these connectors exhibit stable hysteretic response and reliable load transfer mechanisms under repeated loading. Complementing this, Loqman et al. (2018) provided a comprehensive review of the mechanical behavior of steel-concrete composite beams using bolted shear connectors, emphasizing their structural efficiency and potential for prefabricated construction. Further exploration into the reusability of connectors was carried out by Lam et al. (2017), who discussed the latest research on demountable shear connectors. Their study demonstrated that demountable systems offer practical benefits for disassembly and material recycling, without compromising mechanical performance. Qureshi et al. (2011) examined the influence of shear connector spacing and

arrangement, reporting that non-uniform layouts can significantly affect the shear capacity and failure modes of composite beams, underlining the need for optimized connector design in layered systems. In the pursuit of sustainability, Xiong et al. (2024) proposed an innovative steel-concrete composite beam system using novel demountable shear connections. Their findings show improvements in environmental performance, seismic resistance, and ease of construction, making these systems suitable for modern green building practices. Extending this approach, Wang et al. (2025) investigated full-scale steel-concrete beams incorporating recyclable aggregate concrete and bolted shear connectors. Their results confirmed that such systems can deliver both high structural capacity and environmental benefits, supporting the viability of eco-friendly construction materials and methods.

2. FINITE ELEMENT FORMULATION

2.1. Mechanical and material modeling

This study investigates a quintuple-layer functionally graded (FG) beam incorporating shear connectors, as illustrated in Fig. 1.

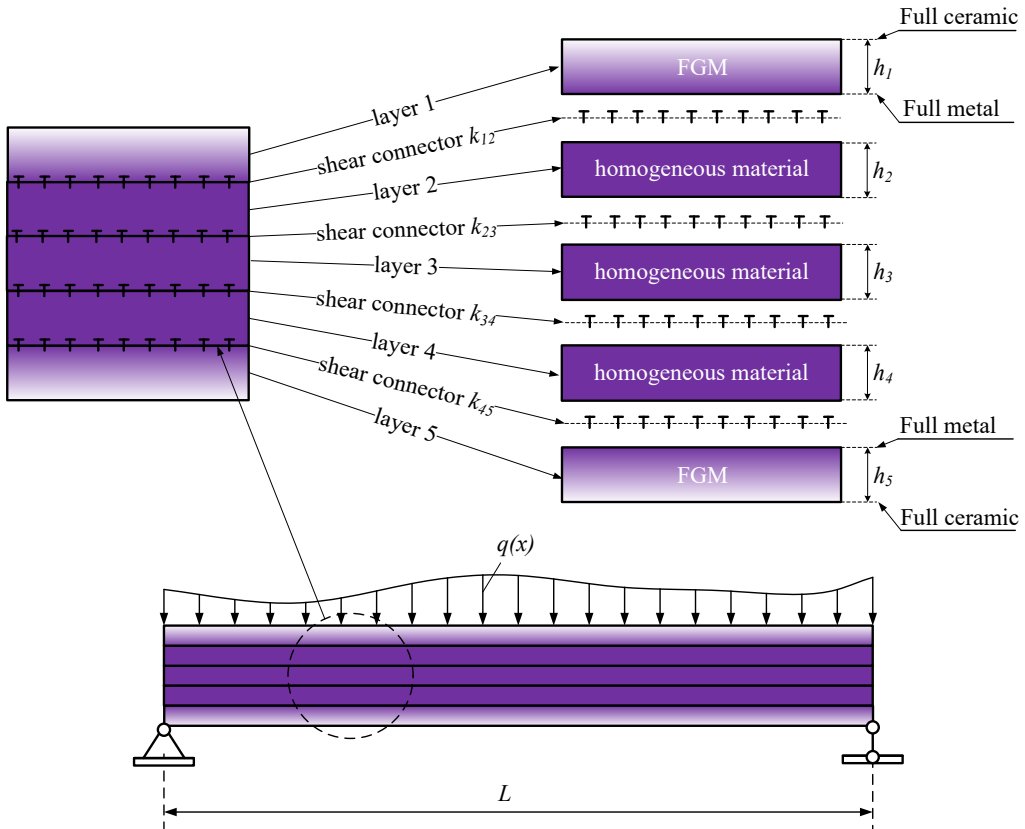


Fig. 1. A detailed model of a quintuple-layer FG beam incorporating shear connectors

Shear connectors are employed to integrate five individual layers (denoted as layers 1 through 5), thereby forming the structural system. In this configuration, layers 2, 3, and 4 are made of a homogeneous metallic material, whereas layers 1 and 5 are constructed using functionally graded materials (FGMs). The volume fraction of metal ($V_m^{(i)}$) and ceramic ($V_c^{(i)}$) in the i -th layer (with $i = 1$ and 5) is defined by a conventional power-law distribution.

$$\text{Layer 1:} \quad \begin{cases} V_c^{(1)} = \left(\frac{1}{2} + \frac{z^{(1)}}{h_1} \right)^{\aleph_1} \\ V_m^{(1)} = 1 - V_c^{(1)} \end{cases} \quad -\frac{h_1}{2} \leq z^{(1)} \leq \frac{h_1}{2} \quad (1)$$

$$\text{Layer 5:} \quad \begin{cases} V_m^{(5)} = \left(\frac{1}{2} + \frac{z^{(5)}}{h_5} \right)^{\aleph_5} \\ V_c^{(5)} = 1 - V_m^{(5)} \end{cases} \quad -\frac{h_5}{2} \leq z^{(5)} \leq \frac{h_5}{2} \quad (2)$$

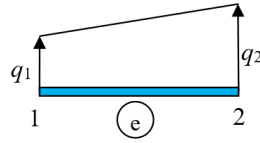
where the thicknesses of layers 1 through 5 are represented by h_1, h_2, h_3, h_4 , and h_5 , respectively, as illustrated in Fig. 1. The volume fraction indices corresponding to layers 1 and 5 are denoted by $\aleph_1$ and $\aleph_5$, while the depth coordinates associated with these layers are given by $z^{(1)}$ and $z^{(5)}$. The symbols m and c refer to the metallic and ceramic constituents, respectively.

The material properties $\Psi^{(i)}$ under consideration, namely Poisson's ratio and Young's modulus, are evaluated using the following expressions, as described in Do et al. (2017), Reddy (2000), and Zenkour (2006).

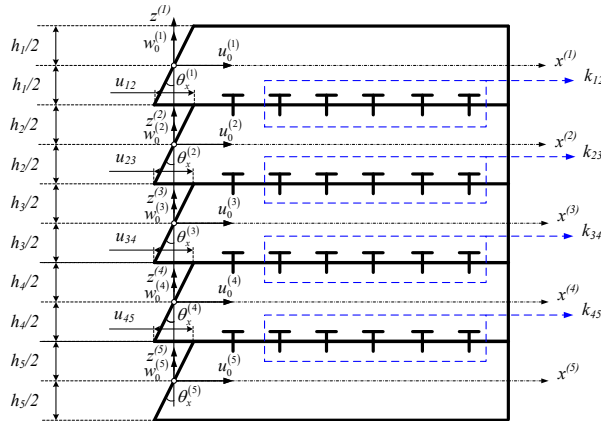
$$\Psi^{(i)}(z^{(i)}) = \Psi_m^{(i)} + (\Psi_c^{(i)} - \Psi_m^{(i)})V_c^{(i)} \quad \text{with } i = 1 \text{ and } 5 \quad (3)$$

2.2. Constitutive relations

Fig. 2 illustrates the finite element model along with the associated degrees of freedom (DOFs) at each node.



(a) Linear interpolation of the displacement field of a two-node beam element



(b) Eleven degrees of freedom (DOFs) at one node

Fig. 2. Schematic of a two-node finite element and its corresponding nodal degrees of freedom (DOFs)

This study is based on several key assumptions, detailed as follows:

- (1) All materials are assumed to behave elastically within their respective limits;
- (2) The self-weight of the shear connectors is neglected;
- (3) External loading is applied perpendicularly to the top surface of the structure, and

(4) No delamination occurs between adjacent layers; only interfacial sliding is permitted.

Here, we employ a two-node beam element. At each layer of the beam, a node has 3 degrees of freedom; with five layers, this results in 15 degrees of freedom at one node. However, the transverse (vertical) degrees of freedom of all layers are identical, so the total number of independent degrees of freedom at a node is reduced to $3 \times 5 - 4 = 11$.

The displacement field at any point along the beam is formulated based on the Timoshenko beam theory, as expressed below (Elishakoff, 2020).

$$\begin{cases} u^{(i)} = u_0^{(i)} + z^{(i)} \cdot \theta_x^{(i)}, \\ w^{(i)} = w_0^{(i)} = w_0, \end{cases} \quad (i = 1-5) \quad (4)$$

In this expression, $u_0^{(i)}$ and $w_0^{(i)}$ denote the displacements along the $x^{(i)}$ and $z^{(i)}$ axes, respectively, at the mid-axis of the individual structural layer. The term $\theta_x^{(i)}$ represents the transverse rotation about the $y^{(i)}$ -axis (normal angle of the beam cross-section around the $y^{(i)}$ -axis, see Fig. 2).

The relative sliding displacement vector at the interface between two adjacent layers can be obtained by applying the following formulations.

$$\mathbf{u}_{\text{relative}} = \begin{Bmatrix} u_{12} \\ u_{23} \\ u_{34} \\ u_{45} \end{Bmatrix} = \begin{Bmatrix} u^{(1)} \left(-\frac{h_1}{2} \right) - u^{(2)} \left(\frac{h_2}{2} \right) \\ u^{(2)} \left(-\frac{h_2}{2} \right) - u^{(3)} \left(\frac{h_3}{2} \right) \\ u^{(3)} \left(-\frac{h_3}{2} \right) - u^{(4)} \left(\frac{h_4}{2} \right) \\ u^{(4)} \left(-\frac{h_4}{2} \right) - u^{(5)} \left(\frac{h_5}{2} \right) \end{Bmatrix} = \begin{Bmatrix} \left(u_0^{(1)} - u_0^{(2)} \right) - \left(\frac{h_1}{2} \cdot \theta_x^{(1)} + \frac{h_2}{2} \cdot \theta_x^{(2)} \right) \\ \left(u_0^{(2)} - u_0^{(3)} \right) - \left(\frac{h_2}{2} \cdot \theta_x^{(2)} + \frac{h_3}{2} \cdot \theta_x^{(3)} \right) \\ \left(u_0^{(3)} - u_0^{(4)} \right) - \left(\frac{h_3}{2} \cdot \theta_x^{(3)} + \frac{h_4}{2} \cdot \theta_x^{(4)} \right) \\ \left(u_0^{(4)} - u_0^{(5)} \right) - \left(\frac{h_4}{2} \cdot \theta_x^{(4)} + \frac{h_5}{2} \cdot \theta_x^{(5)} \right) \end{Bmatrix}. \quad (5)$$

The bending strain distribution of each layer is characterized by the following formulation:

$$\varepsilon_b^{(i)} = \frac{\partial u^{(i)}}{\partial x} = \frac{\partial u_0^{(i)}}{\partial x} + z^{(i)} \cdot \frac{\partial \theta_x^{(i)}}{\partial x} = \varepsilon_m^{(i)} + z^{(i)} \cdot \kappa^{(i)}, \quad (i = 1-5) \quad (6)$$

In this expression, $\varepsilon_m^{(i)}$ represents the membrane bending strain, while $\kappa^{(i)}$ denotes the curvature associated with the i -th component.

The shear strain field corresponding to each layer is defined as follows:

$$\varepsilon_s^{(i)} = \gamma_{xz}^{(i)} = \frac{\partial u^{(i)}}{\partial z^{(i)}} + \frac{\partial w^{(i)}}{\partial x} = \theta_x^{(i)} + \frac{\partial w_0}{\partial x}, \quad (i = 1-5) \quad (7)$$

The stress field within each layer is determined using the following formulation:

$$\begin{cases} \sigma_b^{(1)} = E^{(1)}(z^{(1)}) \cdot \left(\varepsilon_m^{(1)} + z^{(1)} \cdot \kappa^{(1)} \right), \\ \sigma_b^{(2)} = E^{(2)} \cdot \left(\varepsilon_m^{(2)} + z^{(2)} \cdot \kappa^{(2)} \right), \\ \sigma_b^{(3)} = E^{(3)} \cdot \left(\varepsilon_m^{(3)} + z^{(3)} \cdot \kappa^{(3)} \right), \\ \sigma_b^{(4)} = E^{(4)} \cdot \left(\varepsilon_m^{(4)} + z^{(4)} \cdot \kappa^{(4)} \right), \\ \sigma_b^{(5)} = E^{(5)}(z^{(5)}) \cdot \left(\varepsilon_m^{(5)} + z^{(5)} \cdot \kappa^{(5)} \right). \end{cases} \quad (8a)$$

The matrix representation of Eq. (8a) is given as follows:

$$\sigma_b = \mathbf{D}_b \cdot \varepsilon_m + \mathbf{D}_b \cdot \mathbf{Z} \cdot \kappa, \quad (8b)$$

in which $\sigma_b = \{\sigma_b^{(1)} \ \sigma_b^{(2)} \ \sigma_b^{(3)} \ \sigma_b^{(4)} \ \sigma_b^{(5)}\}^T$, $\varepsilon_m = \{\varepsilon_m^{(1)} \ \varepsilon_m^{(2)} \ \varepsilon_m^{(3)} \ \varepsilon_m^{(4)} \ \varepsilon_m^{(5)}\}^T$, and $\kappa = \{\kappa^{(1)} \ \kappa^{(2)} \ \kappa^{(3)} \ \kappa^{(4)} \ \kappa^{(5)}\}^T$ represent the vectors of normal stress, membrane strain, and curvature, respectively. The terms $\mathbf{Z}$ and $\mathbf{D}_b$ denote the coordinate variable and the material property matrices, which are expressed in matrix form as follows:

$$\mathbf{D}_b = \begin{bmatrix} E^{(1)}(z^{(1)}) & 0 & 0 & 0 & 0 \\ 0 & E^{(2)} & 0 & 0 & 0 \\ 0 & 0 & E^{(3)} & 0 & 0 \\ 0 & 0 & 0 & E^{(4)} & 0 \\ 0 & 0 & 0 & 0 & E^{(5)}(z^{(5)}) \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} z^{(1)} & 0 & 0 & 0 & 0 \\ 0 & z^{(2)} & 0 & 0 & 0 \\ 0 & 0 & z^{(3)} & 0 & 0 \\ 0 & 0 & 0 & z^{(4)} & 0 \\ 0 & 0 & 0 & 0 & z^{(5)} \end{bmatrix}. \quad (9)$$

The shear stress distribution within each component is determined as follows:

$$\begin{cases} \tau^{(1)} = \frac{E^{(1)}(z^{(1)})}{2(1+\nu^{(1)}(z^{(1)}))} \cdot \varepsilon_s^{(1)} = \frac{E^{(1)}(z^{(1)})}{2(1+\nu^{(1)}(z^{(1)}))} \cdot \gamma_{xz}^{(1)}, \\ \tau^{(2)} = \frac{E^{(2)}}{2(1+\nu^{(2)})} \cdot \varepsilon_s^{(2)} = \frac{E^{(2)}}{2(1+\nu^{(2)})} \cdot \gamma_{xz}^{(2)}, \\ \tau^{(3)} = \frac{E^{(3)}}{2(1+\nu^{(3)})} \cdot \varepsilon_s^{(3)} = \frac{E^{(3)}}{2(1+\nu^{(3)})} \cdot \gamma_{xz}^{(3)}, \\ \tau^{(4)} = \frac{E^{(4)}}{2(1+\nu^{(4)})} \cdot \varepsilon_s^{(4)} = \frac{E^{(4)}}{2(1+\nu^{(4)})} \cdot \gamma_{xz}^{(4)}, \\ \tau^{(5)} = \frac{E^{(5)}(z^{(5)})}{2(1+\nu^{(5)}(z^{(5)}))} \cdot \varepsilon_s^{(5)} = \frac{E^{(5)}(z^{(5)})}{2(1+\nu^{(5)}(z^{(5)}))} \cdot \gamma_{xz}^{(5)}. \end{cases} \quad (10)$$

As specified in Eq. (10), $\nu^{(i)}$ ($i = 1-5$) denotes the Poisson's ratio corresponding to the i -th layer:

$$\tau = \{\tau^{(1)} \ \tau^{(2)} \ \tau^{(3)} \ \tau^{(4)} \ \tau^{(5)}\}^T = \mathbf{D}_s \cdot \varepsilon_s \quad (11)$$

where $\varepsilon_s = \{\varepsilon_s^{(1)} \ \varepsilon_s^{(2)} \ \varepsilon_s^{(3)} \ \varepsilon_s^{(4)} \ \varepsilon_s^{(5)}\}^T$ denotes the shear strain vector, while $\mathbf{D}_s$ represents the shear material matrix. These quantities are determined as follows:

$$\mathbf{D}_s = \begin{bmatrix} \frac{E^{(1)}(z^{(1)})}{2(1+\nu^{(1)}(z^{(1)}))} & 0 & 0 & 0 & 0 \\ 0 & \frac{E^{(2)}}{2(1+\nu^{(2)})} & 0 & 0 & 0 \\ 0 & 0 & \frac{E^{(3)}}{2(1+\nu^{(3)})} & 0 & 0 \\ 0 & 0 & 0 & \frac{E^{(4)}}{2(1+\nu^{(4)})} & 0 \\ 0 & 0 & 0 & 0 & \frac{E^{(5)}(z^{(5)})}{2(1+\nu^{(5)}(z^{(5)}))} \end{bmatrix} \quad (12)$$

The internal force within each component layer is calculated using the following expression:

$$\begin{cases} N_m^{(1)} = \int_{-h_1/2}^{h_1/2} \sigma_b^{(1)} dz^{(1)} = \int_{-h_1/2}^{h_1/2} E^{(1)}(z^{(1)}) \cdot \varepsilon_b^{(1)} dz^{(1)}, \\ N_m^{(2)} = \int_{-h_2/2}^{h_2/2} \sigma_b^{(2)} dz^{(2)} = \int_{-h_2/2}^{h_2/2} E^{(2)} \cdot \varepsilon_b^{(2)} dz^{(2)}, \\ N_m^{(3)} = \int_{-h_3/2}^{h_3/2} \sigma_b^{(3)} dz^{(3)} = \int_{-h_3/2}^{h_3/2} E^{(3)} \cdot \varepsilon_b^{(3)} dz^{(3)}, \\ N_m^{(4)} = \int_{-h_4/2}^{h_4/2} \sigma_b^{(4)} dz^{(4)} = \int_{-h_4/2}^{h_4/2} E^{(4)} \cdot \varepsilon_b^{(4)} dz^{(4)}, \\ N_m^{(5)} = \int_{-h_5/2}^{h_5/2} \sigma_b^{(5)} dz^{(5)} = \int_{-h_5/2}^{h_5/2} E^{(5)}(z^{(5)}) \cdot \varepsilon_b^{(5)} dz^{(5)}. \end{cases} \quad (13)$$

Now, Eq. (13) can be expressed as follows:

$$\mathbf{N}_m = \mathbf{A} \cdot \boldsymbol{\varepsilon}_m + \mathbf{B} \cdot \boldsymbol{\kappa}, \quad (14)$$

where $\mathbf{N}_m = \{N_m^{(1)} N_m^{(2)} N_m^{(3)} N_m^{(4)} N_m^{(5)}\}^T$ denotes the membrane force vector, while $\mathbf{A}$ and $\mathbf{B}$ are provided in Appendix A.

The bending moment in each component is calculated using the following expression:

$$\begin{cases} M^{(1)} = \int_{-h_1/2}^{h_1/2} \sigma_b^{(1)} \cdot z^{(1)} dz^{(1)} = \int_{-h_1/2}^{h_1/2} E^{(1)}(z^{(1)}) \cdot \varepsilon_b^{(1)} \cdot z^{(1)} dz^{(1)}, \\ M^{(2)} = \int_{-h_2/2}^{h_2/2} \sigma_b^{(2)} \cdot z^{(2)} dz^{(2)} = \int_{-h_2/2}^{h_2/2} E^{(2)} \cdot \varepsilon_b^{(2)} \cdot z^{(2)} dz^{(2)}, \\ M^{(3)} = \int_{-h_3/2}^{h_3/2} \sigma_b^{(3)} \cdot z^{(3)} dz^{(3)} = \int_{-h_3/2}^{h_3/2} E^{(3)} \cdot \varepsilon_b^{(3)} \cdot z^{(3)} dz^{(3)}, \\ M^{(4)} = \int_{-h_4/2}^{h_4/2} \sigma_b^{(4)} \cdot z^{(4)} dz^{(4)} = \int_{-h_4/2}^{h_4/2} E^{(4)} \cdot \varepsilon_b^{(4)} \cdot z^{(4)} dz^{(4)}, \\ M^{(5)} = \int_{-h_5/2}^{h_5/2} \sigma_b^{(5)} \cdot z^{(5)} dz^{(5)} = \int_{-h_5/2}^{h_5/2} E^{(5)}(z^{(5)}) \cdot \varepsilon_b^{(5)} \cdot z^{(5)} dz^{(5)}. \end{cases} \quad (15)$$

The matrix representation of Eq. (15) is given below:

$$\mathbf{M} = \mathbf{B} \cdot \boldsymbol{\varepsilon}_m + \mathbf{D} \cdot \boldsymbol{\kappa}, \quad (16)$$

with $\mathbf{M} = \{M^{(1)} M^{(2)} M^{(3)} M^{(4)} M^{(5)}\}^T$ denoting the bending moment vector, while $\mathbf{D}$ is provided in the Appendix.

The definition of shear force for each layer is given by:

$$\begin{cases} Q^{(1)} = C_{\text{shear}} \int_{-h_1/2}^{h_1/2} \tau^{(1)} dz^{(1)} = C_{\text{shear}} \int_{-h_1/2}^{h_1/2} \frac{E^{(1)}(z^{(1)})}{2(1 + \nu^{(1)}(z^{(1)}))} \cdot \varepsilon_s^{(1)} dz^{(1)}, \\ Q^{(2)} = C_{\text{shear}} \int_{-h_2/2}^{h_2/2} \tau^{(2)} dz^{(2)} = C_{\text{shear}} \int_{-h_2/2}^{h_2/2} \frac{E^{(2)}}{2(1 + \nu^{(2)})} \cdot \varepsilon_s^{(2)} dz^{(2)}, \\ Q^{(3)} = C_{\text{shear}} \int_{-h_3/2}^{h_3/2} \tau^{(3)} dz^{(3)} = C_{\text{shear}} \int_{-h_3/2}^{h_3/2} \frac{E^{(3)}}{2(1 + \nu^{(3)})} \cdot \varepsilon_s^{(3)} dz^{(3)}, \\ Q^{(4)} = C_{\text{shear}} \int_{-h_4/2}^{h_4/2} \tau^{(4)} dz^{(4)} = C_{\text{shear}} \int_{-h_4/2}^{h_4/2} \frac{E^{(4)}}{2(1 + \nu^{(4)})} \cdot \varepsilon_s^{(4)} dz^{(4)}, \\ Q^{(5)} = C_{\text{shear}} \int_{-h_5/2}^{h_5/2} \tau^{(5)} dz^{(5)} = C_{\text{shear}} \int_{-h_5/2}^{h_5/2} \frac{E^{(5)}(z^{(5)})}{2(1 + \nu^{(5)}(z^{(5)}))} \cdot \varepsilon_s^{(5)} dz^{(5)}. \end{cases} \quad (17)$$

In this context, the application of Timoshenko Beam Theory (TBT) requires the inclusion of a shear correction factor, denoted by $C_{\text{shear}} = 5/6$ for this analysis. Accordingly, the shear force can be expressed as follows:

$$\mathbf{Q} = \hat{\mathbf{A}} \cdot \boldsymbol{\varepsilon}_s, \quad (18)$$

with $\hat{\mathbf{A}}$ can be found in Appendix A.

At this stage, Eqs. (14) through (18) can be reformulated in the following matrix form:

$$\begin{Bmatrix} \mathbf{N}_m \\ \mathbf{M} \\ \mathbf{Q} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{A}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_m \\ \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon}_s \end{Bmatrix}. \quad (19)$$

The total strain energy of the beam element, neglecting the effect of shear connectors, is determined using the following expression:

$$U_e = \frac{1}{2} \int_{S_e} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dS_e = \frac{1}{2} \int_{L_e} \left(\int_{-h_i/2}^{h_i/2} \{\boldsymbol{\varepsilon}^{(i)}\}^T \{\boldsymbol{\sigma}^{(i)}\} dz^{(i)} \right) dx, \quad (i = 1-5) \quad (20)$$

in which S_e is the cross-sectional area of the element, and L_e denotes the length of the element. By integrating Eq. (20), the following expression is obtained (Tran et al., 2021).

$$U_e = \frac{1}{2} \int_{L_e} \left(\boldsymbol{\varepsilon}_m^T \mathbf{A} \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_m^T \mathbf{B} \boldsymbol{\kappa} + \boldsymbol{\kappa}^T \mathbf{B} \boldsymbol{\varepsilon}_m + \boldsymbol{\kappa}^T \mathbf{D} \boldsymbol{\kappa} + \boldsymbol{\varepsilon}_s^T \hat{\mathbf{A}} \boldsymbol{\varepsilon}_s \right) dx. \quad (21)$$

In this analysis, a two-node element is utilized, with each node possessing eleven degrees of freedom, as illustrated in Fig. 2. Consequently, the total number of degrees of freedom for each beam element is defined as follows:

$$\mathbf{q}_e = \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}, \quad \mathbf{q}_j = \{u_{0j}^{(1)} \ u_{0j}^{(2)} \ u_{0j}^{(3)} \ u_{0j}^{(4)} \ u_{0j}^{(5)} \ w_{0j} \ \theta_{xj}^{(1)} \ \theta_{xj}^{(2)} \ \theta_{xj}^{(3)} \ \theta_{xj}^{(4)} \ \theta_{xj}^{(5)}\}^T, \quad (j = 1-2) \quad (22)$$

The nodal displacement vector is derived by interpolating the nodal values using the following expression:

$$\mathbf{u} = \{u_0^{(1)} \ u_0^{(2)} \ u_0^{(3)} \ u_0^{(4)} \ u_0^{(5)} \ w_0 \ \theta_x^{(1)} \ \theta_x^{(2)} \ \theta_x^{(3)} \ \theta_x^{(4)} \ \theta_x^{(5)}\}^T = \mathbf{N} \cdot \mathbf{q}_e, \quad (23)$$

where $N_1 = 1 - \zeta$ and $N_2 = 1 + \zeta$ are the Lagrange shape functions, and $\mathbf{N}$ denotes the shape function matrix (refer to Appendix A for details).

Accordingly, the vectors of membrane strain, curvature, and shear strain are expressed as follows:

$$\boldsymbol{\varepsilon}_m = \begin{Bmatrix} \frac{\partial u_0^{(1)}}{\partial x} \\ \frac{\partial u_0^{(2)}}{\partial x} \\ \frac{\partial u_0^{(3)}}{\partial x} \\ \frac{\partial u_0^{(4)}}{\partial x} \\ \frac{\partial u_0^{(5)}}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{u} = \mathbf{B}_1 \cdot \mathbf{q}_e, \quad (24)$$

$$\boldsymbol{\kappa} = \begin{Bmatrix} \frac{\partial \theta_x^{(1)}}{\partial x} \\ \frac{\partial \theta_x^{(2)}}{\partial x} \\ \frac{\partial \theta_x^{(3)}}{\partial x} \\ \frac{\partial \theta_x^{(4)}}{\partial x} \\ \frac{\partial \theta_x^{(5)}}{\partial x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \end{bmatrix} \cdot \mathbf{u} = \mathbf{B}_2 \cdot \mathbf{q}_e, \quad (25)$$

$$\boldsymbol{\varepsilon}_s = \begin{Bmatrix} \theta_x^{(1)} + \frac{\partial w_0}{\partial x} \\ \theta_x^{(2)} + \frac{\partial w_0}{\partial x} \\ \theta_x^{(3)} + \frac{\partial w_0}{\partial x} \\ \theta_x^{(4)} + \frac{\partial w_0}{\partial x} \\ \theta_x^{(5)} + \frac{\partial w_0}{\partial x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{u} = \mathbf{B}_3 \cdot \mathbf{q}_e, \quad (26)$$

where $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ are the matrices of differential shape functions, the detailed expressions of which are provided in Appendix A.

By substituting Eqs. (24)–(26) into Eq. (21), the element stiffness matrix, excluding the effect of shear connectors, can be derived as follows:

$$\mathbf{K}_e^1 = \int_{L_e} \left(\mathbf{B}_1^T \mathbf{A} \mathbf{B}_1 + \mathbf{B}_1^T \mathbf{B} \mathbf{B}_2 + \mathbf{B}_2^T \mathbf{B} \mathbf{B}_1 + \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 + \mathbf{B}_3^T \hat{\mathbf{A}} \mathbf{B}_3 \right) dx. \quad (27)$$

When the effect of shear connectors is taken into account, the strain energy associated with the shear connectors within the elements is calculated as follows (Nam et al., 2019; Nguyen Thai et al., 2021; Van Phung et al., 2022):

$$U_e^{\text{shear connectors}} = \frac{1}{2} \int_{S_e} \mathbf{u}_{\text{relative}}^T \cdot \mathbf{K}_{\text{relative}} \cdot \mathbf{u}_{\text{relative}} dS_e, \quad (28)$$

in which $\mathbf{K}_{\text{relative}} = \begin{bmatrix} k_{12} & 0 & 0 & 0 \\ 0 & k_{23} & 0 & 0 \\ 0 & 0 & k_{34} & 0 \\ 0 & 0 & 0 & k_{45} \end{bmatrix}$ denotes the shear coefficient matrix, and k_{12} represents

the shear stiffness coefficient between layers 1 and 2. Similar definitions apply for the interfaces between layers 2–3, 3–4, and 4–5, as illustrated in Fig. 1. Under external loading, relative sliding is expected to occur between adjacent layers. Consequently, the shear connectors are required to perform mechanical work to counteract the effects of these external forces.

Moreover, the formulation of the relative sliding vector $\mathbf{u}_{\text{relative}}$ is consistent with the structure of Eqs. (24)–(26) and is presented below:

$$\mathbf{u}_{\text{relative}} = \begin{Bmatrix} \left(u_0^{(1)} - u_0^{(2)} \right) - \left(\frac{h_1}{2} \cdot \theta_x^{(1)} + \frac{h_2}{2} \cdot \theta_x^{(2)} \right) \\ \left(u_0^{(2)} - u_0^{(3)} \right) - \left(\frac{h_2}{2} \cdot \theta_x^{(2)} + \frac{h_3}{2} \cdot \theta_x^{(3)} \right) \\ \left(u_0^{(3)} - u_0^{(4)} \right) - \left(\frac{h_3}{2} \cdot \theta_x^{(3)} + \frac{h_4}{2} \cdot \theta_x^{(4)} \right) \\ \left(u_0^{(4)} - u_0^{(5)} \right) - \left(\frac{h_4}{2} \cdot \theta_x^{(4)} + \frac{h_5}{2} \cdot \theta_x^{(5)} \right) \end{Bmatrix} = \mathbf{B}_{\text{relative}} \cdot \mathbf{q}_e, \quad (29)$$

where

$$\mathbf{B}_{\text{relative}} = \sum_{i=1}^2 \begin{bmatrix} N_i & -N_i & 0 & 0 & 0 & 0 & -\frac{h_1}{2}N_i & -\frac{h_2}{2}N_i & 0 & 0 & 0 \\ 0 & N_i & -N_i & 0 & 0 & 0 & 0 & -\frac{h_2}{2}N_i & -\frac{h_3}{2}N_i & 0 & 0 \\ 0 & 0 & N_i & -N_i & 0 & 0 & 0 & 0 & -\frac{h_3}{2}N_i & -\frac{h_4}{2}N_i & 0 \\ 0 & 0 & 0 & N_i & -N_i & 0 & 0 & 0 & 0 & -\frac{h_4}{2}N_i & -\frac{h_5}{2}N_i \end{bmatrix}.$$

By substituting Eq. (29) into Eq. (28), the resulting expression is:

$$\mathbf{K}_e^2 = \int_{L_e} \left(\mathbf{B}_{\text{relative}}^T \cdot \mathbf{K}_{\text{relative}} \cdot \mathbf{B}_{\text{relative}} \right) dx. \quad (30)$$

By integrating Eqs. (27) and (30), the element stiffness matrix corresponding to the shear connectors can be formulated as follows:

$$\mathbf{K}_e = \mathbf{K}_e^1 + \mathbf{K}_e^2. \quad (31)$$

Consider a distributed external load $q(x)$ acting on the upper surface of the beam, as illustrated in Fig. 1. By applying the principle of virtual work, the external work done by this load can be expressed as follows (Tran et al., 2021):

$$A_e = \int_{L_e} \mathbf{u}^T \cdot \mathbf{p} dx, \quad (32)$$

where $\mathbf{p}$ is the vector of nodal loads, which is determined as follows:

$$\mathbf{p} = \{0 \ 0 \ 0 \ 0 \ 0 \ q(x) \ 0 \ 0 \ 0 \ 0 \ 0\}^T. \quad (33)$$

Substituting Eq. (23) into Eq. (32) yields the following result:

$$A_e = \mathbf{q}_e^T \left(\int_{L_e} \mathbf{N}^T \cdot \mathbf{p} dx \right) = \mathbf{q}_e^T \cdot \mathbf{F}_e. \quad (34)$$

As a result, the nodal force vector corresponding to the element can be formulated as:

$$\mathbf{F}_e = \int_{L_e} \mathbf{N}^T \cdot \mathbf{p} dx. \quad (35)$$

The global system, comprising the stiffness matrix $\mathbf{K}$ and force vector $\mathbf{F}$, is constructed through the assembly of individual element stiffness matrices $\mathbf{K}_e$ and corresponding element force vectors $\mathbf{F}_e$. The governing equation used to solve the static problem is given as follows:

$$\mathbf{K} \cdot \mathbf{Q} = \mathbf{F}, \quad (36)$$

in which $\mathbf{Q}$ denotes the global displacement vector.

3. NUMERICAL EVALUATION AND DISCUSSION

3.1. Verifications

In this section, validation examples related to bending behavior are presented to verify the accuracy of the computational program developed in MATLAB. Specifically, a quintuple-layer functionally graded (FG) beam with shear connectors is examined. The beam possesses the following geometric properties: length $L = 6$ m, width $b_i = 0.4$ m ($i = 1 \div 5$), and thickness $h_i = 0.04$ m ($i = 1 \div 5$). Layers 1 and 5 are made of functionally graded Al/Al₂O₃ materials, while layers 2, 3, and 4 are composed of a homogeneous metal (Al) with the material parameters specified in Table 1.

Table 1. Material mechanical characteristics

Material component	Elastic Modulus E (GPa)	Poisson's Ratio ν
Aluminum (Al)	70	0.30
Alumina (Al ₂ O ₃)	380	0.30

In this example, parameter $\aleph_1 \rightarrow \infty$ is selected in Eq. (1), while Eq. (2) employs parameter $\aleph_5 = 0$. The shear coefficient is denoted by $k_{12} = k_{23} = k_{34} = k_{45} \rightarrow \infty$, indicating the absence of relative sliding between the layers (refer to Fig. 3).

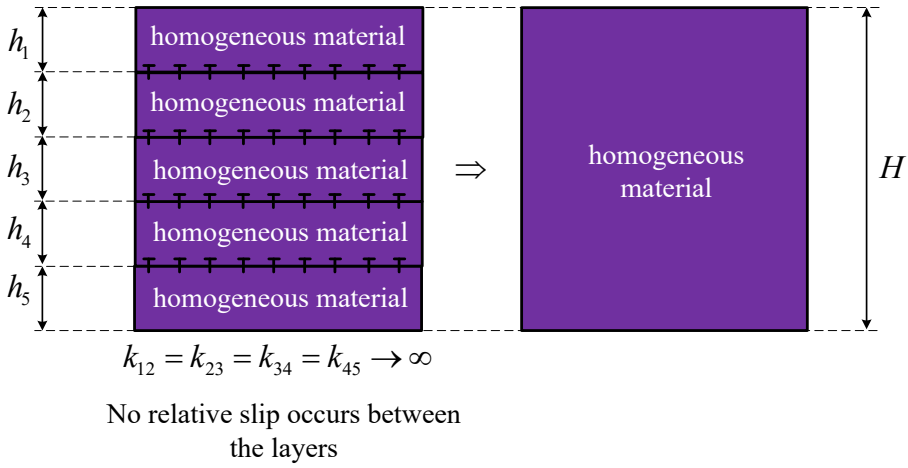


Fig. 3. Transformation of a quintuple-layer FG beam with shear connectors into an equivalent homogeneous beam

The current quintuple-layer FG beam with shear connectors is now idealized as a homogeneous aluminum beam with a width of $B = b_i = 0.4$ m ($i = 1 \div 5$), a depth of $H = \sum_{i=1}^5 h_i = 0.20$ m, and a length $L = 6$ m.

- Illustration 1: Uniformly distributed load $q(x) = 1.5 \times 10^4$ N/m.

Illustration 1-1: C-C

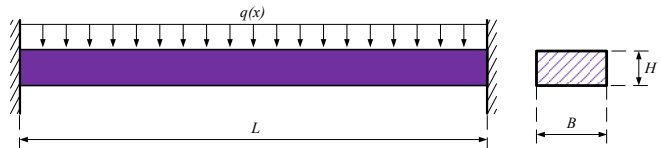


Illustration 1-2: C-S

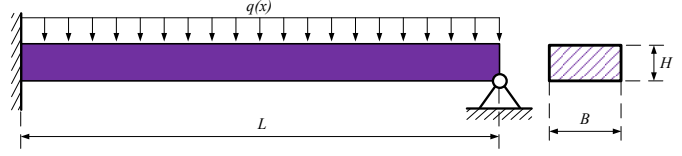


Illustration 1-3: S-S

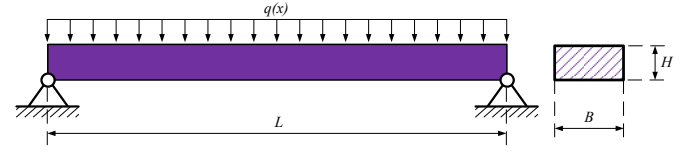
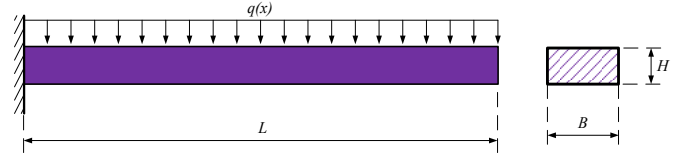


Illustration 1-4: C-F



- Illustration 2: Concentrated moment $M = 3 \times 10^4 \text{ N}\cdot\text{m}$.

Illustration 2-1: C-C

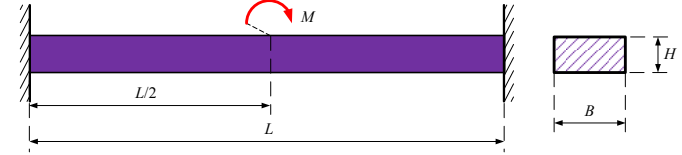


Illustration 2-2: C-S

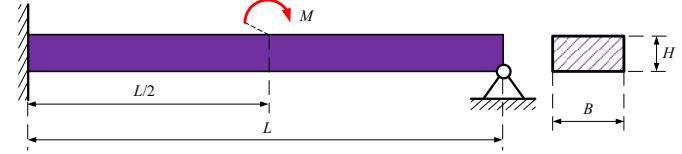


Illustration 2-3: S-S

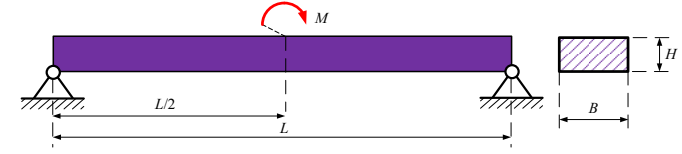
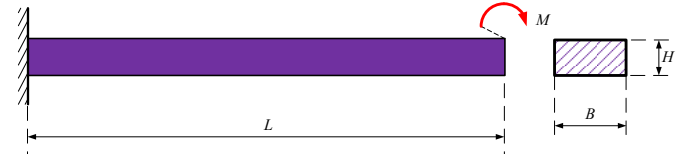


Illustration 2-4: C-F



As illustrated by Table 2, the approximate results are compared with exact solutions from the literature.

In the previous cases, the comparison was made for $L/H = 30$. Next, the authors consider a beam for a simply supported beam (S-S) subjected to a concentrated load at the midpoint of the beam, $P = 2 \times 10^4 \text{ N/m}$, as shown in Fig. 4.

The discrepancies between the present results and the exact solutions depend on the L/H ratio, as shown in Table 3.

Across all models, the results show a very small discrepancy between the present results and the exact solutions. Therefore, the computational program developed in the MATLAB environment can be used to investigate and develop more complex problems.

Table 2. Comparison of maximum deflection results in some cases is considered

Illustration	Maximum deflection results (m)		Difference (%)
	This paper	Exact solution (Bansal, 2015; Khurmi, 2006)	
1-1: C-C	2.7448×10^{-3}	$w_{\max} = \frac{1}{384} \cdot \frac{q(x)L^4}{E_m I} = 2.7120 \times 10^{-3}$	1.21
1-2: C-S	0.0056771	$w_{\max} = \frac{(-15 + \sqrt{33})^2(21 + 5\sqrt{33})}{786432} \cdot \frac{q(x)L^4}{E_m I} = 0.0056405$	0.65
1-3: S-S	0.013576	$w_{\max} = \frac{5}{384} \cdot \frac{q(x)L^4}{E_m I} = 0.013560$	0.12
1-4: C-F	0.130301	$w_{\max} = \frac{1}{8} \cdot \frac{q(x)L^4}{E_m I} = 0.130178$	0.09
2-1: C-C	2.67932×10^{-3}	$w_{\max} = \frac{1}{216} \cdot \frac{ML^2}{E_m I} = 2.67857 \times 10^{-4}$	0.03
2-2: C-S	4.65005×10^{-4}	$w(x = L/2) = \frac{1}{128} \cdot \frac{ML^2}{E_m I} = 4.52008 \times 10^{-4}$	2.87
2-3: S-S	4.71522×10^{-4}	$w_{\max} = \frac{\sqrt{3}}{216} \cdot \frac{ML^2}{E_m I} = 4.63942 \times 10^{-4}$	1.64
2-4: C-F	0.029021	$w_{\max} = \frac{1}{2} \cdot \frac{ML^2}{E_m I} = 0.028928$	0.32

Note that $I = BH^3/12$ (Bansal, 2015; Khurmi, 2006).

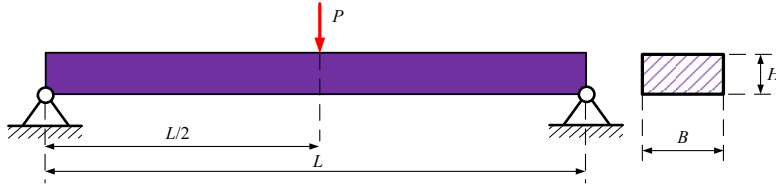


Fig. 4. Simply supported beam model under a concentrated load at mid-span

Table 3. Comparison of the deflection of the beam as a function of the L/H ratio

L/H ($L = 6$ m is constant)	Maximum deflection results (m)		Difference (%)
	This paper	Exact solution (Bansal, 2015; Khurmi, 2006)	
		$w_{\max} = \frac{1}{48} \cdot \frac{PL^3}{E_m I}$	
5	0.2538×10^{-4}	0.2232×10^{-4}	13.73
10	0.1844×10^{-3}	0.1785×10^{-3}	3.30
15	0.6109×10^{-3}	0.6026×10^{-3}	1.36
20	0.00143829	0.00142857	0.68

3.2. Parameter study

3.2.1. Influence of the stiffness of shear connectors

Consider a beam with the same geometrical and material parameters as in the verification problem, along with the same boundary conditions and load as in Illustration 1-4, with one end clamped and the other end free. The parameters used are the same as those in the verification example. Here, the interlayer slip coefficients are defined as $k_{12} = k_{23} = k_{34} = k_{45} = \alpha E_m$. By

varying the values of α , the corresponding deflection responses of the beam are obtained, as illustrated in Fig. 5.

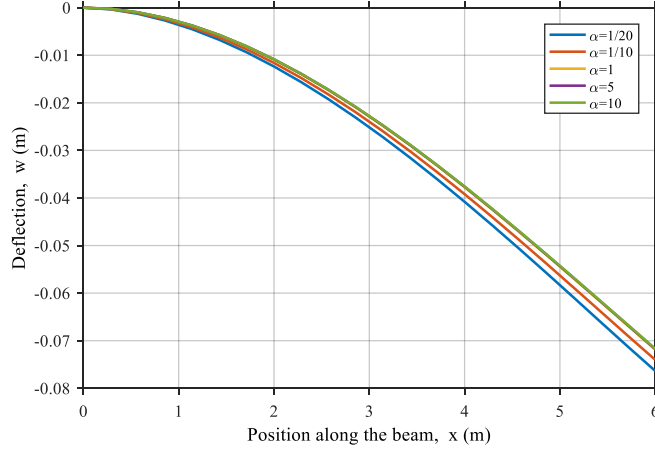


Fig. 5. Beam deflection under different stiffnesses of shear connectors

It is observed that as the α coefficient increases, the stiffness of the shear connectors increases. The overall stiffness of the structure also increases, thus absorbing more energy from the external force. As a result, the deflection response decreases as shown in Fig. 5.

3.2.2. Influence of boundary conditions

Finally, the authors investigate the influence of boundary conditions on the static response of the beam. The beam parameters are identical to those used in the verification example, with the stiffness coefficients of the shear connectors set as $k_{12} = k_{23} = k_{34} = k_{45} = \frac{1}{10}E_m$. A uniformly distributed load $q(x) = 2 \times 10^4$ N/m. Four types of boundary conditions are: clamped-clamped (CC); simply supported-simply supported (SS); clamped-simply supported (CS); clamped-free (CF). Fig. 6 illustrates the deflection responses of the beam under various boundary conditions.

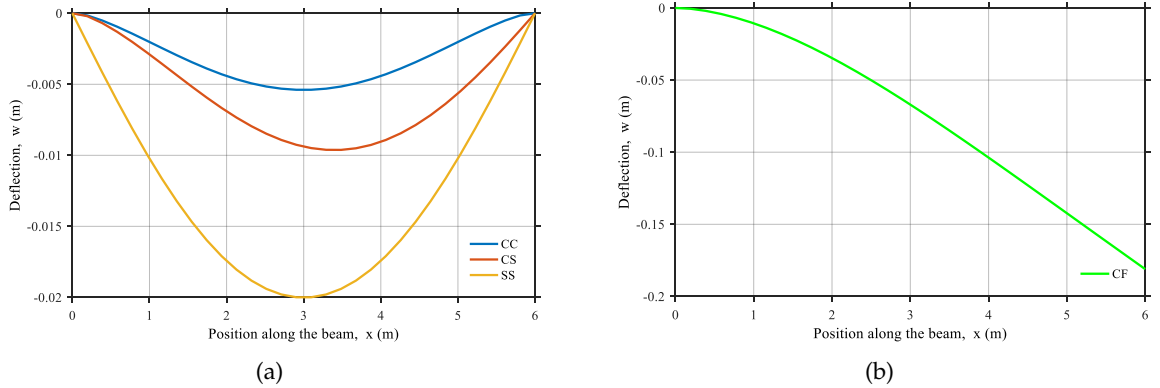


Fig. 6. Deflection of the beam under different boundary conditions

It is found that with the same input parameters, the fewer degrees of freedom the beam has, the stiffer the structure becomes. This leads to a decrease in the beam's displacement response.

4. CONCLUSIONS

This study has developed a finite element model for analyzing the static response of a quintuple-layer composite beam composed of functionally graded materials (FGMs) and equipped with shear connectors. The model is formulated based on the first-order shear deformation theory (FSDT) of Timoshenko and implemented using a two-node beam element with eleven degrees of freedom per node to accurately capture interlayer sliding effects. The incorporation of shear connectors in the model provides a realistic representation of the interaction between layers, which significantly influences the overall stiffness and deformation characteristics of the beam. The proposed numerical method has been successfully implemented in the MATLAB environment and validated through comparisons with benchmark results. The findings confirm the reliability and efficiency of the model in simulating complex layered beam systems, thereby offering a robust tool for structural analysis. The outcomes of this work provide a foundation for extending the model to more advanced applications, such as free vibration analysis, dynamic response studies, and optimization of connector layout. Future research may also focus on integrating nonlinear material behavior, thermal effects, and damage evolution to broaden the applicability of the model in engineering practice.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CREDIT AUTHOR STATEMENT

Phung Van Minh: *Conceptualization, Methodology, Formal analysis, Software, Validation, Visualization, Writing - original draft, Writing - review & editing.* Ninh Xuan Chung: *Investigation, Data curation, Formal analysis, Visualization, Writing - original draft.* Tran Van Ke: *Supervision, Project administration, Funding acquisition, Resources, Validation, Writing - review & editing.* Ta Duc Tam: *Investigation, Data curation, Resources, Visualization.*

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APPENDIX A.

The matrices **A**, **B**, **D**, and **Â** are given as follows:

$$\mathbf{A} = \begin{bmatrix} \int_{-h_1/2}^{h_1/2} E^{(1)}(z^{(1)}) dz^{(1)} & 0 & 0 & 0 & 0 \\ 0 & E^{(2)}h_2 & 0 & 0 & 0 \\ 0 & 0 & E^{(3)}h_3 & 0 & 0 \\ 0 & 0 & 0 & E^{(4)}h_4 & 0 \\ 0 & 0 & 0 & 0 & \int_{-h_5/2}^{h_5/2} E^{(5)}(z^{(5)}) dz^{(5)} \end{bmatrix}, \quad (\text{A.1})$$

$$\mathbf{B} = \begin{bmatrix} \int_{-h_1/2}^{h_1/2} E^{(1)}(z^{(1)})z^{(1)} dz^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \int_{-h_5/2}^{h_5/2} E^{(5)}(z^{(5)})z^{(5)} dz^{(5)} & 0 \end{bmatrix}, \quad (\text{A.2})$$

$$\mathbf{D} = \begin{bmatrix} \int_{-h_1/2}^{h_1/2} E^{(1)}(z^{(1)})(z^{(1)})^2 dz^{(1)} & 0 & 0 & 0 & 0 \\ 0 & E^{(2)} \frac{h_2^3}{12} & 0 & 0 & 0 \\ 0 & 0 & E^{(3)} \frac{h_3^3}{12} & 0 & 0 \\ 0 & 0 & 0 & E^{(4)} \frac{h_4^3}{12} & 0 \\ 0 & 0 & 0 & 0 & \int_{-h_5/2}^{h_5/2} E^{(5)}(z^{(5)})(z^{(5)})^2 dz^{(5)} \end{bmatrix}, \quad (\text{A.3})$$

$$\hat{\mathbf{A}} = \begin{bmatrix} C_{\text{shear}} \int_{-h_1/2}^{h_1/2} \frac{E^{(1)}(z^{(1)})}{2(1+\nu^{(1)}(z^{(1)}))} dz^{(1)} & 0 & 0 & 0 & 0 \\ 0 & C_{\text{shear}} \frac{E^{(2)}h_2}{2(1+\nu^{(2)})} & 0 & 0 & 0 \\ 0 & 0 & C_{\text{shear}} \frac{E^{(3)}h_3}{2(1+\nu^{(3)})} & 0 & 0 \\ 0 & 0 & 0 & C_{\text{shear}} \frac{E^{(4)}h_4}{2(1+\nu^{(4)})} & 0 \\ 0 & 0 & 0 & 0 & C_{\text{shear}} \int_{-h_5/2}^{h_5/2} \frac{E^{(5)}(z^{(5)})}{2(1+\nu^{(5)}(z^{(5)}))} dz^{(5)} \end{bmatrix}. \quad (\text{A.4})$$